# Transverse Parton Distribution Functions at Next-To-Next-To-Leading Order

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# Abstract

We present a perturbative calculation of all parton-to-parton transverse parton distribution functions up to next-to-next-to-leading order based on a gauge invariant operator definition. We demonstrate for the first time that such a definition works beyond the first non-trivial order. We extract the coefficient functions relevant for a next-to-next-to-next-to-leading logarithmic $q_T$ resummation in a large class of processes at hadron colliders.

# Auszug

Wir bestimmen die transversalen Partonverteilungsfunktionen bis zur nächst-nächst-führenden störungstheoretischen Ordnung für alle Parton-Parton Übergänge. Hierzu nutzen wir eine eichinvariante Operatordefinition und zeigen zum ersten Mal, dass diese auch über die erste nicht triviale Ordnung hinaus anwendbar ist. Mit unserer Rechnung bestimmen wir außerdem die Koeffizientenfunktionen, welche für die Resummation der nächst-nächst-nächst-führenden $q_T$ Logarithmen in vielen Hadron-Hadron-Streuprozessen benötigt werden.

# Contents

# 1 Introduction

Particle physics aims to describe the properties of matter and its interactions down to very small distances. The currently best established theory of elementary particle physics is the Standard Model. The discovery of the Higgs boson candidate by the ATLAS and CMS collaborations at CERN last summer [1, 2] is another important fortification of this model, since it predicted the existence of that boson. The important role of the Higgs field is to generate masses of elementary particles. For the development of the relevant mechanism [3, 4], this years Nobel price in physics has been awarded.

While these developments date back nearly 50 years ago, it took a long time and very much effort at many frontiers to eventually discover the Higgs boson candidate. Despite its large mass, one main complication has been the fact that the experimental signal of its existence is hidden below a huge background, which is caused by the Standard Model physics outside the Higgs sector. While this has clearly been a huge challenge on the experimental side, it also challenged theoretical physicists to predict both the signal and the background to high accuracy.

To measure the properties of the Higgs boson candidate, to determine the parameters in the Standard Model to high accuracy, to find signals of New Physics and for many other tasks, dedicated experiments and high accurate calculations are therefore highly demanded.

To obtain the required high precision in the theoretical predictions, observables have to be determined to high perturbative order in the coupling constants - in particular in the strong coupling constant of Quantum Chromodynamics (QCD), as it is much larger than the other couplings. However, a fixed order calculation is usually not sufficient. We often have to deal with physical problems that involve several separate mass scales. The ratios of these scales can lead to large logarithms, which spoil the convergence of the perturbative expansion. They therefore have to be accounted for to all orders in perturbation theory. This is referred to as resummation.

Our focus is on the resummation of Sudakov logarithms of transverse momentum. They are, for example, important when differential cross sections are considered at small transverse momentum. For their resummation, many complementary methods exist, possessing various advantages and disadvantages. A frequently used, intuitive and flexible approach is the Monte Carlo parton shower. Unfortunately, this approach usually only provides the leading logarithmic (LL) and parts of the next-to- (N) LL contributions. Higher precision can be achieved by (semi)-analytical methods. While in this way many observables have been determined to NLL, only few of them have been studied up to N$^3$LL precision.

We will determine a set of process independent functions, which are relevant for the N$^3$LL transverse momentum resummation of the differential cross sections of all hadron collider processes, in which a color neutral final state of high invariant mass is created. We will apply the framework of [5, 6]. It is based on the soft-collinear effective theory (SCET) of QCD and allows the derivation of all order factorization theorems for the processes of interest. After factorization, the different pieces can be consistently determined in fixed order perturbation theory and the logarithms resummed by solving the renormalization-group (RG) evolution equations.

The various pieces in these factorization theorems can be related to the parts of the well established factorization theorems and resummation methods pioneered several decades ago by [7]. The extension of the factorization to all perturbative orders and the high order determination of the independent pieces have been a long standing problem. We will determine the parton-to-parton versions of the process independent functions, appearing in the factorization theorems of [5, 6], up to next-to-next-to-leading order (NNLO) in perturbation theory. These are the anomaly coefficients and the transverse parton distribution functions (TPDFs). They are obtained from an explicit operator definition, which is directly implied by the factorization theorem. Thus, our calculation is the first one, which accomplished their NNLO calculation directly from first principles.

The TPDFs are generalizations of usual collinear PDFs. Both describe the distribution of partons inside hadrons. While the PDFs only resolve the partons' energy fraction, the TPDFs moreover resolve their transverse momentum. This is necessary for example to consistently describe the distribution of the resolved final state at small transverse momentum ($q_T$), since the latter is generated from recoiling against the initial state radiation. If this transverse scale resides in the perturbative regime, the TPDFs can be expressed as Mellin convolutions of normal PDFs with perturbative matching kernels. We extract these kernels up to NNLO. In terms of them, we provide process independent functions, relevant for N$^3$LL $q_T$-resummation. As a byproduct, we confirm the process specific $\mathcal{H}^{(2)}$ coefficients of [8, 9] and reextract the $\alpha_s^2$ contributions of the DGLAP splitting kernels.

While the high order $q_T$ resummation is an important application of the transverse factorization and the TPDFs, they are also relevant for many other aspects in QCD. For example, they are important to describe spin or azimuthal related observables as well as to understand the spin structure of the proton and other hadrons.

Despite the special kinematics, which introduces a special integral kernel, a main complication in our calculation arises from the presence of light-cone (LC) propagators. They do not only increase the number of independent denominators, but also give rise to rapidity singularities, which are not regulated in dimensional regularization. In our perturbative calculation, we therefore use the additional, analytic regulator as suggested in [10]. For the final results this regulator can be consistently removed and gives rise to the collinear anomaly discussed in [5].

Assuming standard text book knowledge of the reader on Quantum field theories, we start our discussion in chapter 2.1 by introducing basic ideas and concepts of effective field theories (EFTs) in general. We then introduce in chapter 2.2 the soft-collinear effective theory (SCET) of QCD, which is the EFT appropriate for the kinematic situation, we aim to describe. SCET is applied in chapter 3 to rederive factorization theorems for Drell-Yan-

(DY) and Higgs-production. There, we also provide details about the whole framework and about the relevance of our calculation. It, furthermore, leads us directly to the operator definition of the naive (n) TPDFs. Chapter 4 outlines the structure of our perturbative calculation of these objects. The successive chapters give details to the calculation steps and provide intermediate results: Chapter 5 addresses the NLO contribution, chapter 6 the virtual-real (VR) NNLO contribution and chapter 7 the real-real (RR) NNLO contribution. In chapter 8, we show explicitly, how the analytic and the dimensional regulators can be consistently removed and the final results can be extracted. These results are presented in chapters 8.3 and 8.4. Thereafter, we discuss several checks, we performed to confirm the correctness of our results. We conclude in chapter 9.

# 2 Effective Field Theories

In this chapter, we will first discuss some general aspects about effective field theories (EFT). Later on we will consider soft-collinear effective theory. Throughout this chapter, we will follow [11].

## 2.1 Local EFT

As discussed in section 1, perturbative calculations of processes, which involve several disparate mass scales, suffer from the presence of large logarithms of ratios of these scales, which can spoil the convergence of fixed order perturbation theory. These logarithms have to be resummed to all orders. A very generic and systematic approach to deal with multi-scale problems in quantum field theories are EFTs. The basic idea is to systematically separate physics at different length scales. In the following subsection we will start with an example illustrating the construction of an effective field theory.

### 2.1.1 Illustrative Example

Let us consider a process involving a large mass scale $M$ for energies $E \ll M$. Introducing a cutoff $\Lambda$ between those two scales, we can divide a quantum field $\phi$ in a part $\phi_H$ containing the Fourier modes with energy $E_H > \Lambda$ and a part $\phi_L$ containing the remaining small energy modes, i.e.

$$\phi(x) = \phi_L(x) + \phi_H(x)\,. \tag{2.1}$$

If we are solely interested in physics at low energies $E \ll \Lambda$, we will only need to consider correlation functions between the fields $\phi_L(x)$. Within the path integral formalism the correlation function

$$\langle 0|T\phi_L(x_1)\ldots\phi_L(x_n)|0\rangle = \frac{1}{Z[0]}(-i\delta_{J_L(x_1)})\ldots(-i\delta_{J_L(x_n)})Z[J_L]\Big|_{J_L=0} \tag{2.2}$$

is obtained by the generating functional of the theory

$$Z[J_L] = \int\mathcal{D}\phi_H\mathcal{D}\phi_L \exp\left\{iS(\phi_L,\phi_H) + i\int \mathrm{d}^d x J_L(x)\phi_L(x)\right\}. \tag{2.3}$$

The action $S(\phi_L, \phi_H)$ is related to the Lagrangian $\mathcal{L}(x)$ as

$$S(\phi_L, \phi_H) = \int \mathrm{d}^d x \mathcal{L}(x) \,. \tag{2.4}$$

As we are only interested in the correlation functions of the light fields $\phi_L(x)$, there was no need to include sources for the field $\phi_H$. We can even go a step further and perform the path integral over this field. With the Wilsonian effective action $S_\Lambda(\phi_L)$ related to the action of the full theory via

$$\exp\Big\{iS_\Lambda(\phi_L)\Big\} = \int \mathcal{D}\phi_H \exp\Big\{iS(\phi_L, \phi_H)\Big\} \,, \tag{2.5}$$

the generating functional can be written as

$$Z[J_L] = \int \mathcal{D}\phi_L \exp\Big\{iS_\Lambda(\phi_L) + i\textstyle\int \mathrm{d}^d x J_L(x)\phi_L(x)\Big\} \,, \tag{2.6}$$

which is free of an explicit dependence on $\phi_H$. This field has been 'integrated out'. The Wilsonian effective action $S_\Lambda(\phi_L)$ depends on the cut off scale $\Lambda$. As the modes of energies $E_H > \Lambda$ have been removed, it is non-local at scales $\Delta x \sim 1/\Lambda$.

However, as by assumption $E \ll \Lambda$, we can perform an operator product expansion (OPE) for the effective action in terms of local operators composed from light fields and their derivatives:

$$S_\Lambda(\phi_L) = \int \mathrm{d}^d x \mathcal{L}_\Lambda^{\mathrm{eff}}(x) \,, \tag{2.7}$$

with the effective Lagrangian

$$\mathcal{L}_\Lambda^{\mathrm{eff}}(x) = \sum_i c_i(M) O_i(\phi_L) \,. \tag{2.8}$$

Each term is given by a local operator $O_i$ multiplied by the corresponding coupling $c_i$, usually referred to as Wilson coefficient. All operators allowed by the symmetries of the theory will appear in this sum. Because an operator can contain arbitrarily many fields, the sum has infinitely many terms. By construction the operators $O_i$ do not know about the large mass scale $M$. Thus in our case the only mass scale relevant for them is $E$. For a given mass dimension $\omega_i = d + \gamma_i$ of $O_i$, we therefore expect $O_i \sim E^{\omega_i}$. Moreover, we write

$$c_i(M) = \frac{C_i(M)}{M^{\gamma_i}} \tag{2.9}$$

with dimensionless coefficients $C_i$. As $M$ is the only fundamental mass scale relevant for them, $C_i = \mathcal{O}(1)$ is expected by the hypotheses of naturalness. This implies however, that the contribution of operators with $\omega_i > d$ is suppressed by $(E/M)^{\gamma_i}$ w.r.t. operators with $\omega_j = d$. For $E \ll M$ this is a strong suppression. Hence, in that energy range, we will only need to include operators up to a maximal dimension $\omega_{\max}$. The exact choice of this maximal dimension depends on the required precision. In the cases relevant for us $\omega_{\max} = d$. However, in a more general case also subleading operators can be important. One example is, if the separation of the two scales is not very large. Another example is, if one considers the Standard model as an effective field theory of a parent theory, then it is exactly those operators which contain interesting physical information about physics beyond the Standard Model.

In any case, the important point is that it is sufficient to consider only the first couple of terms in the infinite sum in eqn. (2.8) and that the corresponding loss in precision can be estimated by the ratio of scales. For this reason the effective theory is a useful tool, which allows the separation of the high from the low energy scales.

Operators with $\omega_i < d$ in eqn. (2.8) should be forbidden by the symmetries of the effective theory. If such operators have been present in the full theory, they should not affect the low energy effective theory as the physics of such large energies has been integrated out.

Let us also comment on the dependence of eqn. (2.8) on the scale $\Lambda$ we used for separating the high and low energy modes. If this scale is sufficiently smaller then the high mass scale $M$ and sufficiently larger than other relevant physics scales, then a small change $\delta\Lambda$ of $\Lambda$ will not change the form of this equation, as the symmetries dictating the operators appearing in the sum are still the same. However, the degrees of freedom in $[\Lambda, \Lambda + \delta\Lambda]$ will be moved between the Wilson coefficients $C_i$ and the fields $\phi_L$, which by construction of the EFT depend on $\Lambda$. This implies a running of those coefficients with $\Lambda$.

The arguments presented here for a scalar field with only a single mass scale, can be generalized for other fields and more complex theories of several scales and fields. Furthermore, the way the fields are split in several components is usually done differently from a hard cut off. However, the basic concepts are the same as discussed above.

Let us now for a general case discuss, how the objects in eqn. (2.8) can be obtained from the full theory and why this separation can be very useful in practice.

### 2.1.2 Matching and Renormalization

If the parent theory $\mathcal{L}$ is known, the Wilson coefficients can be determined by 'matching' the effective theory to the full theory. To this end, one compares the matrix elements

$$\langle f|\mathcal{L}|i\rangle \overset{!}{=} \langle f|\mathcal{L}^{\text{eff}}|i\rangle = \sum_{i|_{\omega_i=d}} C_i(\mu)\langle f|O_i(\mu)|i\rangle + \mathcal{O}\big(E/M\big) \tag{2.10}$$

for various final and initial states $\langle f|$ and $|i\rangle$ at each order in perturbation theory, and by this fixes the coefficients order by order. Since the coefficients are independent of the considered states, once fixed, they can be used to determine further matrix elements. Note that for simplicity we assume here and in the following that only operators with $\omega_i = d$ have to be considered. However, including operators with higher dimensions is straightforward.

This approach to match the effective theory on the full one is only sensible if there exists a scale $\mu$ above which the full theory is weakly coupled, such that perturbation theory is a sensible approach. Then in contrast to the amplitudes themselves, the Wilson coefficients are insensitive to any infrared (IR) physics. This also implies that if required, we are free to choose arbitrary IR regulators. Those will cancel between the two theories for the results of $C_i$.

Let us consider a toy example with a single operator and the coupling $\alpha = \frac{g^2}{4\pi}$. Be $M$ the large scale and $p^2$ the IR regulator in form of an off-shell momentum, then for the two sides of eqn. (2.10) to order $\alpha^1$ we might find

$$1 + \alpha A \log \tfrac{M^2}{p^2} = C(\mu)\big(1 + \alpha A \log \tfrac{\mu^2}{p^2}\big) + \mathcal{O}(\alpha^2)\,, \tag{2.11}$$

with some constant $A$, which implies the Wilson coefficient as

$$C(\mu) = 1 + \alpha A \log \frac{M^2}{\mu^2} + \mathcal{O}(\alpha^2)\,. \tag{2.12}$$

Besides illustrating how the IR regulator does not enter the Wilson coefficients, this example also shows, how by the construction of the effective field theory the two mass scales of the high and low energy physics become separated. While in the full theory both scales are intertwined in the argument of the logarithm, in the effective theory, the high scale only enters the Wilson coefficient and the small scale only the operator. This is one reason why EFT are very useful when dealing with multi-scale problems or situations where a mass scale is significantly larger than the energy region of consideration.

Just as the bare matrix elements of the full theory, the bare matrix elements of the effective theory will have divergences, which require a regularization. Usually dimensional regularization is chosen. The matrix elements of the individual operators of the effective theory often have higher poles than the full theory matrix element. In some cases, they might even need additional regularization, e.g. by analytic regulators. However, all additional poles have to cancel in the combination of all parts of the effective theory. Later in this work we will encounter examples where analytic regulators are needed, now, however, we focus on a situation where the regularization of the full theory is sufficient for the effective theory, too.

As also the effective theory is a quantum field theory, it can be renormalized. This is done by operator renormalization, which has the form

$$\langle O_i(\mu)\rangle_b = \sum_j Z^O_{ij}(\mu)\langle O_j(\mu)\rangle_r \tag{2.13}$$

with the renormalization matrix $Z_{O,ij} = \delta_{ij} + \mathcal{O}(\alpha)$, which mix the operators under renormalization. As the Lagrangian should reflect the same theory, no matter if expressed in terms of bare or renormalized quantities, the RHS of eqn. (2.10), once expressed by bare functions and once by renormalized functions, should be equal. This implies for the Wilson coefficients

$$C_i^r(\mu) = \sum_j C_j^b(\mu) Z_{ji}^O(\mu)\,, \tag{2.14}$$

as can easily be confirmed by comparing the RHS of eqn. (2.10) for bare and renormalized expressions. Note that in the last equation the renormalized function is on the left, not on the right hand side.

The renormalized quantities and renormalization matrix are renormalization scale and scheme dependent. An important remark is in place. In general the operator renormalization of eqn. (2.13) can mix all operators, including operators of different mass dimension. This would be very unpractical, as then operators of higher mass dimension would be reintroduced. However, if one uses a mass independent regularization scheme, like dimensional regularization, which we will use throughout this work in the $\overline{\text{MS}}$ variant, the mixing is only between operators of the same mass dimension.

### 2.1.3 RG-Improved Perturbation Theory

Let $\{O_i(\mu)\}$ with $i = 1, \ldots n$ be a basis of operators of dimension $\omega_i$ allowed by the symmetries, we then can formulate the renormalization scale independence of the RHS of eqn. (2.10) as

$$\frac{d}{d\log\mu} \sum_{i=1}^{n} C_i(\mu) \langle O_i(\mu)\rangle = 0\,, \tag{2.15}$$

where here and in the following we consider renormalized operators and suppress the corresponding label. By the product rule of differentiation, the left hand side (LHS) will contain derivatives of the operators with respect to $\log\mu$. As the operators $O_{i\ldots n}(\mu)$ form a basis, those derivatives can be expressed in terms of this basis again, i.e.

$$\frac{d}{d\log\mu} \langle O_i(\mu)\rangle = -\sum_{i=j}^{n} \gamma_{ij}(\mu) \langle O_j(\mu)\rangle\,, \tag{2.16}$$

where we introduced the dimensionless parameters $\gamma_{ij}$, which are usually called anomalous dimensions. They encode how the operators change under infinitesimal scale variations and hence are free of large logarithms. Moreover, they depend on the renormalization scale only via the coupling $\alpha(\mu)$. From the last two equations and the fact that the operators are linearly independent and the external states can be chosen arbitrarily, we find

$$\frac{d}{d\log\mu} C_j(\mu) - \sum_{i=1}^{n} C_i(\mu) \gamma_{ij}(\mu) = 0\,, \tag{2.17}$$

the renormalization group (RG) equations of the Wilson coefficients. Using a matrix notation and the fact that the anomalous dimensions depend on $\mu$ only via $\alpha(\mu)$, this can be rewritten as

$$\frac{d}{d\alpha(\mu)}\vec{C}(\mu) = \frac{\hat{\gamma}^T(\alpha(\mu))}{\beta(\alpha(\mu))}\vec{C}(\mu)\,, \tag{2.18}$$

where we used the $\beta$-function of the considered theory, $\beta = d\alpha(\mu)/d\log\mu$. This equation can be solved, leading to

$$\vec{C}(\mu) = T_{\alpha\downarrow}\exp\left[\int_{\alpha(M)}^{\alpha(\mu)} \mathrm{d}\alpha \frac{\hat{\gamma}^T(\alpha)}{\beta(\alpha)}\right]\vec{C}(M)\,, \tag{2.19}$$

which connects the values of the Wilson coefficients at two different mass scales. The exponential of the matrix is defined as its Taylor expansion. The terms therein are ordered by $T_{\alpha\downarrow}$ such that factors with larger $\alpha$ stand to the left of factors with smaller $\alpha$.

The last equation implies that providing $\vec{C}(M_i)$ as initial conditions at some scale $M_i$, we can obtain the corresponding value at any other sensible scale. In that equation we already specified $M_i = M$, where $M$ is the large scale of the full theory. The reason is that by construction of the effective field theory the logarithms contained in $\vec{C}(\mu)$ are $\log\frac{M}{\mu}$, as has been sketched around eqn. (2.12). While in the region $\mu \ll M$, in which we will be interested, these logarithms are large, they are very small for $\mu \sim M$. Hence, at this scale the Wilson coefficients can be determined sensibly in fixed order perturbation theory.

Once obtained at this high scale, solving eqn. (2.19) for $\vec{C}(\mu)$ at a low scale automatically resums the large logarithms to all orders in perturbation theory. Then we can choose $\mu$ of the order of a typical mass scale relevant for the matrix elements of the operators of the EFT, such that the logarithms of scale ratios appearing there are small and the fixed order perturbative calculation is free of large logarithms. This is possible as the high and low mass scale are separated into the Wilson coefficient and the effective operators by construction of the EFT.

The points outlined above are great virtues of EFT and RG-improved perturbation theory. It is for their reason that we will apply their methods.

The effective theory of a form as discussed in this section, which will be relevant for our further discussion, is the effective theory obtained by integrating out the heavy and high virtuality modes around the top mass scale $m_t$ to obtain an effective coupling of gluons to the Higgs boson.

This will be relevant for our consideration of the factorization of Higgs production at hadron colliders. Apart from the discussion done in this section, we will not discuss the construction of this effective theories in more detail. A corresponding discussion can, however, be found in [12, 13], for example.

After arriving at the level of this effective theory, we will go yet a step further and consider the effective theory of its QCD part when integrating out high virtuality gluons. To respect the specific kinematic situation of the production of heavy final states with small transverse momentum at hadron colliders, soft-collinear effective theory (SCET) is the suitable effective theory. It will be discussed in the following section.

In addition to Higgs production, we will also consider the production of other color neutral final states of high-invariant mass and small transverse momentum, e.g. the production of a Drell-Yan pair. Also in these cases we will integrate out the high virtuality gluons and use SCET as the relevant effective theory of QCD.

## 2.2 Soft-Collinear Effective Theory

Soft-Collinear Effective Theory (SCET) is an effective theory of QCD. As such it reproduces the low energy physics of QCD and allows for the separation of physics at different scales. SCET is more complicated than the type of EFT discussed earlier. The main additional complication it has to resolve is the separation of modes of light particles with low virtuality which have nevertheless large energy from those which have small energy. We then have to take non-local effects into account and therefore generalize the concept of local OPE.

SCET was introduced in [14–16] for the first time. For our discussion, however, we continue following [11].

The explicit construction steps of SCET depend on the kinematic situation under consideration. Hence, we will specify the kinematics relevant for our consideration in the next subsection. After that we illustrate, how the effective theory is constructed.

### 2.2.1 Kinematics

For explicitness let us consider a specific kinematic situation. At a hadron collider, we consider two colliding beams of hadrons. The direction of one being $n^\mu$, the direction of the other $\bar{n}^\mu$. In the final state, we consider a high energetic photon $\gamma^*$ with high virtuality $M^2$ which will subsequently decay into a Drell-Yan pair. The photon carries transverse momentum $p_T$ w.r.t. the colliding beams.

In addition to the photon, there will be the beam remnants in the final state. Many of the particles therein will have low energy and momentum. They will be referred to as soft. Most of the remaining particles will have momenta approximately collinear to one of the beam directions. Even though their virtuality is usually small, they can have large energy. To such particles we will refer to as collinear if they move approximately in the $n^\mu$ direction and as anti-collinear if they move approximately in the $\bar{n}^\mu$ direction. As those remnants have to balance the transverse momentum of the photon, their transverse momentum can be of similar size.

In our considerations, we are interested in the phenomenologically important region, where $\Lambda_{\mathrm{QCD}} \ll p_T^2 \ll M^2$. Then by the ratio of the two mass scales of the problem we can define the small parameter

$$\lambda = \frac{q_T}{M} \,. \tag{2.20}$$

This small parameter will be the expansion parameter for the construction of the effective field theory, similar to the large mass scale in the last section.

To make our discussion more precise and clear, it is convenient to use light-cone coordinates which we define as follows. Given the light-cone (LC) vectors $n^\mu$ and $\bar{n}^\mu$ fulfilling $n^2, \bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$, each vector $k^\mu$ can be decomposed in terms of these 4-vectors and

| momentum mode (m) | $(p_+, p_-, p_\perp)/M$ | $p^2/M^2$ |
|---|---|---|
| hard ($h$) | $\sim (1, 1, 1)$ | $\sim 1$ |
| collinear ($c$) | $\sim (\lambda^2, 1, \lambda)$ | $\sim \lambda^2$ |
| anti-collinear ($\bar{c}$) | $\sim (1, \lambda^2, \lambda)$ | $\sim \lambda^2$ |
| soft ($s$) | $\sim (\lambda, \lambda, \lambda)$ | $\sim \lambda^2$ |
| ultra-soft ($us$) | $\sim (\lambda^2, \lambda^2, \lambda^2)$ | $\sim \lambda^4$ |

Table 2.1: Scaling of the different momentum modes.

a component $k_\perp^\mu$ perpendicular to them

$$k^\mu = k_- \frac{n^\mu}{2} + k_+ \frac{\bar{n}^\mu}{2} + k_\perp^\mu = k_-^\mu + k_+^\mu + k_\perp^\mu \,, \tag{2.21}$$

where we used the shorthand notation

$$\bar{n} \cdot k = k_- \,, \quad n \cdot k = k_+ \,., \tag{2.22}$$

for vector product containing the light-cone momenta. The vector can then also be written as $k = (k_+, k_-, k_\perp)$. To the Lorentz vector $k_\perp$ we also define its Euclidean partner $k_T$. For the scalar products of two vectors the decomposition (2.21) implies

$$k \cdot l = \frac{k_- l_+}{2} + \frac{k_+ l_-}{2} - k_T \cdot l_T \,. \tag{2.23}$$

In our case, the two light-cone vectors are $n^\mu$ and $\bar{n}^\mu$, the directions of the colliding beams. In the lab frame they are given by $n^\mu = (1, 0, 0, 1)^\mu$ and $\bar{n}^\mu = (1, 0, 0, -1)^\mu$. With this last piece of notation and $\lambda$ from eqn. (2.20), we can be more precise about the different modes of momentum relevant for us. They are given in table 2.1. Depending on the scaling of the three components with $\lambda$, we distinguish ultra-soft ($us$), soft ($s$), collinear ($c$), anti-collinear ($\bar{c}$) and hard ($h$) modes. While for the first four modes at least two components are suppressed by $\lambda$, for the hard mode no component is suppressed.

While the fields within each region interact normally, interactions between fields of different regions are constrained, due to the different scaling. Interactions between ultra-soft and other fields have to always include at least two of the fields of the same non-ultra-soft region, because the ultra-soft modes can not compensate for the large momentum component. Interactions between $c$ and $\bar{c}$ fields are not possible, with the exception of the creation of a hard field from a $c$ and a $\bar{c}$ field.

One might wonder if in addition to the modes in table 2.1, further modes with different scaling should be considered. For the kinematic situation in which we are interested and with the regularization we will choose in our later calculation it has been argued in [5] that such further modes do not contribute. Various of these modes would not contribute at leading power in $\lambda$ or are not allowed or relevant kinematically. The other of these modes would lead to scaleless integrals which vanish in dimensional regularization.

For our specific kinematic situation with our specific choice of regularization, following [5] it will turn out, that at leading order in $\lambda$ furthermore neither the ultra-soft nor the soft modes listed in table 2.1 will contribute. This is observed, only after these modes have been decoupled from the (anti)-collinear modes at leading order in $\lambda$. These decouplings are important steps for SCET factorization theorems. Therefore, we will show for the ultra-soft modes, how this decoupling can be achieved. While the basic ideas to show the decoupling of the soft modes are similar, the corresponding discussion is more involved. Therefore, for simplicity, we will not discuss it and only provide the corresponding result.

### 2.2.2 Construction of the EFT

We start from the QCD Lagrangian and split, analogously to eqn. (2.1), each quark and gluon field into different subfields. These subfields are chosen to have momenta scaling as hard, collinear, anti-collinear, ultra-soft or soft. This can be achieved by restricting the Fourier integral to momenta with the appropriate scaling. In a first step all modes with high virtuality are integrated out. This can be done in the standard way discussed in section 2.1. For our consideration, we are then left with a form of QCD composed of (anti)-collinear and (ultra)-soft fields, which can couple to the high invariant mass modes, including non-QCD modes as the energetic photon, via effective operators. In the rest of this section, we will discuss, how the effective theory of the remaining soft, collinear and anti-collinear fields looks like.

Recall that the collinear and anti-collinear fields do not interact directly with each other. Furthermore, by exchanging the notion of the LC-vectors $n$ and $\bar{n}$, we exactly exchange the notion of collinear and anti-collinear fields. Because by this exchange the consideration of anti-collinear fields is directly implied, we simplify our discussion by considering a theory of ultra-soft and collinear fields only.

Among all the possible terms, we are only interested in those of lowest order in the expansion parameter $\lambda$, since the additional contributions are suppressed by powers of this parameter. To achieve a consistent expansion in $\lambda$, we have to know the scaling of the various fields. For each field, the scaling can be obtained from the scaling of the corresponding propagators. While the quark propagator is given by

$$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{ip\cdot(x-y)} \frac{i}{p^2+i\epsilon} (\not{p}_+ + \not{p}_- + \not{p}_\perp)\,, \tag{2.24}$$

the gluon propagator is given by

$$\langle 0|TA^\mu(x)A^\nu(y)|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{ip\cdot(x-y)} \frac{i}{p^2+i\epsilon} \Big( -g^{\mu\nu} + (1-a)\frac{p^\mu p^\nu}{p^2}\Big)\,. \tag{2.25}$$

For each field mode the momentum scales as given in table 2.1. Inserting these scalings in the last two equations and using $\mathrm{d}^4p = \frac{1}{2}\mathrm{d}p_+\mathrm{d}p_-\mathrm{d}p_\perp^2$, we find the scalings of the fields as written in table 2.2. To find the scaling, only the largest terms have to be considered. For

| field | $\psi_{us}$ | $\psi_s$ | $\psi_c$ | $A_{us}$ | $A_s$ | $A_c$ |
|---|---|---|---|---|---|---|
| scaling | $\sim \lambda^3$ | $\sim \lambda^{3/2}$ | $\sim \lambda$ | $\sim (\lambda^2, \lambda^2, \lambda^2)$ | $\sim (\lambda, \lambda, \lambda)$ | $\sim (\lambda^2, 1, \lambda)$ |

Table 2.2: Scaling of the different field modes. $\psi$ stands for a quark, $A$ for a gluon.

$\psi_c$, these come from the term with $\not{p}_- \sim 1$ and for $A_c^\mu$ from the term with $p^\mu$. Thus we find the scaling of the gluon field to be the same as the corresponding momenta.

The contribution of soft and ultra-soft quark fields is power suppressed w.r.t. the (anti)-collinear quark field. It will therefore not be relevant in our discussion. The ultra-soft gluon field is of similar size as the small component of the $c$ fields. We will discuss its contribution in the next section. As explained earlier, we will not include the soft field in our explicit discussion. Therefore, it will not be present in the following. For the rest of this section, we will discuss the collinear quark field. It can be decomposed into two subfields

$$\psi_c(x) = \xi(x) + \eta(x)\,, \text{ with} \tag{2.26}$$

$$\not{n}\xi(x) = 0\,, \quad \not{\bar{n}}\eta(x) = 0\,, \tag{2.27}$$

where we used the projection operators $\frac{1}{4}\not{n}\not{\bar{n}}$ and $\frac{1}{4}\not{\bar{n}}\not{n}$ to define

$$\xi(x) = \frac{\not{n}\not{\bar{n}}}{4}\psi_c(x)\,, \tag{2.28}$$

$$\eta(x) = \frac{\not{\bar{n}}\not{n}}{4}\psi_c(x)\,. \tag{2.29}$$

This is, we decompose the full 4-component spinor into two spinors with only two independent components. For each of the components we can consider the propagator (2.24) multiplied from each side with the relevant projector. The largest contribution $\not{p}_- = p_- \not{n}$ will drop out for the terms involving $\eta$. We then find the scalings $\xi \sim \lambda$ and $\eta \sim \lambda^2$. In the next step, we will among others remove the latter component. To simplify our discussion here, we do not include the power suppressed mass term in our discussion, even though this could be done straightforwardly.

Let us express the Dirac Lagrangian in terms of $\xi$, $\eta$, and $A = A_s + A_c$

$$\begin{aligned}\mathcal{L} &= \bar{\psi}_c i\not{D}\psi_c = (\bar{\xi}+\bar{\eta})i\not{D}(\xi+\eta) \\ &= \bar{\xi}\frac{\not{\bar{n}}}{2}in\cdot D\xi + \bar{\eta}\frac{\not{n}}{2}i\bar{n}\cdot D\eta + \bar{\xi}i\not{D}_\perp\eta + \bar{\eta}i\not{D}_\perp\xi\,.\end{aligned} \tag{2.30}$$

In the last step we decomposed $D^\mu$ in LC coordinates and used eqn. (2.27). The soft quark field we did not include, as its contribution is power suppressed. We now remove the field

$\eta$ by solving its equation of motion

$$\frac{\not{n}}{2} i\bar{n}\cdot D\eta + i\not{D}_\perp \xi = 0 \quad \Rightarrow \quad \eta = -\frac{\not{\bar{n}}}{2}\frac{1}{i\bar{n}\cdot D + i\epsilon} i\not{D}_\perp \xi \,. \tag{2.31}$$

Removing $\eta$ in this way will by the equation of motion also remove the sum of the second and fourth term of the Lagrangian. The removal of $\eta$ can also be done more rigorous by performing the path integral of the $\eta$ field. Effectively the result is however the same. Note that due to the presence of the covariant derivative in the denominator, the result is highly non-local. To formulate it in a more sensible way, we introduce the Wilson line

$$W(x) = P\exp\left[ig_s\int_{-\infty}^{0}\mathrm{d}s\;\bar{n}\cdot A(x+s\bar{n})\right], \tag{2.32}$$

which is the solution to the differential equation $i\bar{n}\cdot D\, W(x) = 0$. In the equation above, $P$ is the path ordering. In the expansion of the exponent, it places fields with higher values of $us$ to the left to fields with lower values. The Hermitian conjugate $W^\dagger(x)$ of the Wilson line is given by an analogous equation with $i$ replaced by $-i$ and $P$ by $\bar{P}$, which orders the field in the opposite way. Among others, the Wilson line and his Hermitian conjugate have the important properties

$$W(x)W^\dagger(x) = 1\,, \quad W^\dagger(x)W(x) = 1 \quad \text{and} \tag{2.33}$$

$$W^\dagger(x)\, i\bar{n}\cdot D\, W(x) = i\bar{n}\cdot\partial\,. \tag{2.34}$$

The last property implies

$$W^\dagger(x)\,\frac{1}{i\bar{n}\cdot D + i\epsilon}\,W(x) = \frac{1}{i\bar{n}\cdot\partial + i\epsilon}\,. \tag{2.35}$$

An inverse ordinary derivative acting on a function can be identified as an integral over this function. Then we can write the equation for $\eta$ as

$$\begin{aligned}\eta(x) &= W(x)\frac{\not{\bar{n}}}{2}\frac{(-1)}{i\bar{n}\cdot\partial + i\epsilon}(W^\dagger i\not{D}_\perp\xi)(x) \\ &= W(x)\frac{\not{\bar{n}}}{2} i\int_{-\infty}^{0}\mathrm{d}s\big(W^\dagger i\not{D}_\perp\xi\big)(x+s\bar{n})\,.\end{aligned} \tag{2.36}$$

With this expression we can rewrite the third term of the effective, non-local Lagrangian and receive

$$\mathcal{L}(x) = \bar{\xi}(x)\frac{\not{\bar{n}}}{2} in\cdot D\xi(x) - \big(\bar{\xi}i\overleftarrow{\not{D}}_\perp W\big)(x)\frac{\not{\bar{n}}}{2} i\int_{-\infty}^{0}\mathrm{d}s\big(W^\dagger i\not{D}_\perp\xi\big)(x+s\bar{n})\,, \tag{2.37}$$

where we also used integration by parts to let $\overleftarrow{\not{D}}_\perp$ act to the left. Although we proceeded already far, there are several things left to do. Most importantly, we have to guarantee proper power counting and drop subleading contributions from the Lagrangian. Moreover, we want to disentangle the remaining contributions of soft gluons from the collinear fields.

### 2.2.3 Decoupling of Ultra-Soft Modes

Regarding the power counting, we observe from table 2.1 and 2.2 that all components of the ultra-soft gluon field, but $n \cdot A_{us}$, are subleading in $\lambda$ w.r.t. the corresponding components of $A_c$ and the momentum. Hence, $n \cdot A_{us}$ is the only ultra-soft component we have to keep while others have to be dropped. This yields

$$\begin{aligned} in\cdot D &= in\cdot D_c + g_s n\cdot A_{us}\,, \\ \not{D}_\perp &= \not{D}_{c,\perp} + \mathcal{O}(\lambda^2)\,, \\ W(x) &= W_c(x) + \mathcal{O}(\lambda^2)\,, \end{aligned} \tag{2.38}$$

where the label $c$ reminds us that only collinear gluon fields are present. Since $\bar{n}\cdot A_c(x)$ is not suppressed by $\lambda$, also $W_c(x)$ still contains the infinite sum of this field component.

To remove subleading contributions, the remaining ultra-soft field in the interactions with collinear fields has to be multi-pole expanded. This is, we write

$$A_{us}(x) = A_{us}(x_-) + \mathcal{O}(\lambda)\,. \tag{2.39}$$

The power suppressed terms are of the form $(x_\perp \cdot \partial_\perp + x_+ \cdot \partial_+ + \ldots)A_{us}(x_-)$ with the derivatives being evaluated before $x$ is set to $x_-$. The form and the need of this power expansion follows from considering

$$S_{\text{int}} \supset g_s\!\int\!\mathrm{d}^4x\; \bar{\xi}(x)\, n\cdot A_{us}(x)\, \xi(x) \sim \int\!\mathrm{d}^4x\; e^{ix\cdot(\bar{p}_c - p_{us} - p_c)}\bar{\xi}(0)\, n\cdot A_{us}(0)\, \xi(0)\,. \tag{2.40}$$

Since the combined momentum in the exponent scales as a collinear momentum, the leading contribution to the action arises for $x = (x_+, x_-, x_\perp) \sim (1, \lambda^{-2}, \lambda^{-1})$. Then the phase space factor for the ultra-soft momentum can be Taylor expanded in $\lambda$ as

$$e^{-ip_{us}\cdot x} = e^{-ip_{us+}x_-}\big(1 - ip_{us\perp}\cdot x_\perp - ip_{us-}\cdot x_+ + \ldots\big)\,, \tag{2.41}$$

which leads to eqn. (2.39) and the power suppression of the corrections as discussed there. Taken together we obtain the leading power ($\lambda^0$) SCET Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{SCET}}(x) =& \bar{\xi}(x)\frac{\not{\bar{n}}}{2} in\cdot D_c \xi(x) + \bar{\xi}(x)\frac{\not{\bar{n}}}{2} g_s n\cdot A_{us}(x_-)\xi(x) \\ &- \big(\bar{\xi} i\overleftarrow{\not{D}}_{c\perp} W_c\big)(x)\frac{\not{\bar{n}}}{2} i\int_{-\infty}^{0}\mathrm{d}s\big(W_c^\dagger i\not{D}_{c\perp}\xi\big)(x+s\bar{n}) + \mathcal{O}(\lambda)\,. \end{aligned} \tag{2.42}$$

In contrast to Lagrangians of usual EFTs the SCET Lagrangian contains terms with arbitrarily many fields at leading order in the expansion parameter. These enter through the Wilson line which is - just as the corresponding component of the gluon field - unsuppressed in $\lambda$. Moreover, the Lagrangian contains integrals along the path of the LC vector $\bar{n}$, which appear explicitly and in the exponent of the Wilson lines. Hence, the Lagrangian is non-local.

At this order in $\lambda$, the only remaining interaction with ultra-soft fields is through the second term of the Lagrangian containing $n\cdot A_{us}(x_-)$. We will now remove this interaction through a field redefinition. To this end we introduce the the ultra-soft Wilson line

$$Y_n(x_-) = P\exp\left[ig_s\int_{-\infty}^{0}\mathrm{d}s\ \bar{n}\cdot A_{us}(x_- + sn)\right] \tag{2.43}$$

along the light-cone direction $n$. With $Y_n$ and its Hermitian conjugate $Y_n^\dagger$, we can define the new collinear fields

$$\begin{aligned} \xi^{(0)}(x) &= Y_n^\dagger(x_-)\xi(x)\,, \\ A_c^{(0)\mu}(x) &= Y_n^\dagger(x_-)A_c^\mu(x)Y_n(x_-)\,. \end{aligned} \tag{2.44}$$

Since $Y_n(x_-)$ is the solution to the differential equation $\left(in\cdot D_{us}Y_n^\dagger\right) = 0$, this implies $Y_n^\dagger(x_-)in\cdot D_{us}Y_n = in\cdot\partial$. Thus in terms of the new fields the Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{\mathrm{SCET}}(x) =&\bar{\xi}^{(0)}(x)in\cdot D_c^{(0)}\xi^{(0)}(x) \\ &+\left(\bar{\xi}^{(0)}i\not{D}_{c\perp}^{(0)}W_c^{(0)}\right)(x)\frac{\not{\bar{n}}}{2}i\int_{-\infty}^{0}\mathrm{d}s\left(W_c^{(0)\dagger}i\not{D}_{c\perp}^{(0)}\xi^{(0)}\right)(x+s\bar{n}) + \mathcal{O}(\lambda)\,, \end{aligned} \tag{2.45}$$

where only collinear modes are left. Note that the decoupling only holds at leading order in the expansion parameter $\lambda$. The decoupling of ultra-soft modes from the leading order collinear Lagrangian is the basis of SCET factorization of collinear and ultra-soft modes. We will discuss its implications for the two examples of our interest in section 3. Also in several other kinematic situations, where different versions of SCET must be applied, an appropriate decoupling allows proofs of factorization theorems.

### 2.2.4 Residual Symmetry

Before turning to the derivation of the factorization theorems relevant for us, let us comment on the residual symmetry of the SCET Lagrangian.

During the derivation of the SCET Lagrangian, we split the fields in different momentum modes and integrated out some of them, furthermore, we introduced explicit dependencies on the two vectors $n$ and $\bar{n}$. Which subclass of the symmetries of QCD does then remain for the SCET Lagrangian?

The two symmetries which are affected are gauge invariance and Lorentz symmetry. Let us consider gauge symmetry first.

Order by order in $\lambda$, the SCET Lagrangian is invariant under collinear $U_c^{(0)}(x)$ and ultra-soft $U_{us}(x)$ gauge transformations. They preserve the scaling properties of the fields, i.e. their derivatives scale like a collinear or ultra-soft momentum, respectively. They can be defined by restricting their Fourier decomposition to collinear or ultra-soft modes, respectively. Under collinear gauge transformations, the ultra-soft fields and ultra-soft Wilson line do not transform, while the collinear fields transform as

$$\begin{aligned}\xi^{(0)}(x) &\to U_c^{(0)}(x)\xi^{(0)}(x)\,,\\ A_c^{(0)\mu}(x) &\to U_c^{(0)}(x)A_c^{(0)\mu}(x)U_c^{(0)\dagger}(x)+\frac{i}{g_s}U_c^{(0)}(x)\big[\partial^\mu,U_c^{(0)\dagger}(x)\big]\,.\end{aligned}\tag{2.46}$$

The collinear Wilson line and the $c$ covariant derivative acting on the $c$ quark field transform like a collinear quark field.

Under ultra-soft gauge transformations the collinear fields and Wilson line do no transform. The ultra-soft fields transform as

$$\begin{aligned}q_{us}(x) &\to U_{us}(x)q_{us}(x)\,,\\ A_{us}^\mu(x) &\to U_{us}(x)A_{us}^\mu(x)U_{us}^\dagger(x)+\frac{i}{g_s}U_{us}(x)\big[\partial^\mu,U_{us}^\dagger(x)\big]\,.\end{aligned}\tag{2.47}$$

The ultra-soft Wilson line, the ultra-soft covariant derivative acting on the ultra-soft quark field and the remaining part of a heavy quark field in HQET transform as a ultra-soft quark field.

Note that in the literature the transformations are often discussed on the level of the collinear fields $\xi(x)$ and $A_c^\mu$ before they have been redefined via eqn. (2.44). Those fields transform under ultra-soft gauge transformations and the collinear gauge transformations of $n\cdot A_c^\mu$ involve ultra-soft gluon fields. It was the combination with the ultra-soft Wilson line which not only decoupled the ultra-soft gluons from the collinear Lagrangian but also made the new collinear fields insensitive to ultra-soft gauge transformations.

With the explicit form of the gauge transformations discussed above, it is easy to check that the SCET Lagrangian (2.45) is invariant under collinear and ultra-soft gauge transformations. We also observe that we can build gauge invariant building blocks by combining the fields and covariant derivatives with the corresponding Wilson line as

$$\chi(x)=W_c^{(0)\dagger}(x)\xi^{(0)}(x)\,,\tag{2.48}$$

$$\mathcal{A}^\mu(x)=\frac{1}{g_s}\big[W_c^{(0)\dagger}\big(D_c^{(0)\mu}W_c^{(0)}\big)\big](x)\,.\tag{2.49}$$

Correspondingly, gauge invariant operators for the ultra-soft field can be constructed. In the second term the derivative only acts on $W_c^{(0)}(x)$, as is indicated by the additional brackets. In a gauge with $\bar{n}\cdot A_c^{(0)}(x)=0$, we have $W_c^{(0)}(x)=1$ and the building blocks simplify to $\xi^{(0)}(x)$ and $A_c^{(0)\mu}(x)$.

In the next section it will be useful to use these gauge invariant building blocks to write operator definitions for physical observables in a manifestly gauge invariant form.

Let us turn to Lorentz invariance. Due to the presence of the light-cone vectors, the

| Type | $n^\mu$ | $\bar{n}^\mu$ | constraint |
|---|---|---|---|
| I | $n^\mu \to n^\mu + \epsilon_\perp^\mu$ | $\bar{n}^\mu \to \bar{n}^\mu$ | $\epsilon_\perp^\mu \sim \lambda$ |
| II | $n^\mu \to n^\mu$ | $\bar{n}^\mu \to \bar{n}^\mu + e_\perp^\mu$ | $e_\perp^\mu \sim \lambda$ |
| III | $n^\mu \to \alpha^{-1} n^\mu$ | $\bar{n}^\mu \to \alpha \bar{n}^\mu$ | $\alpha \sim 1$ |

Table 2.3: Reparameterization invariance of the SCET Lagrangian.

SCET Lagrangian is not explicitly Lorentz invariant, even though this symmetry is still present in SCET. As consequence of this invariance, operators in SCET must be invariant under the three classes of infinitesimal transformations of the LC vectors given in table 2.3. Those transformations correspond to two different transverse and a longitudinal boost, respectively. They imply constraints on Wilson coefficients of SCET operators. We however will not extend this discussion here. More detail and references can be found in [17], which is also the reference we followed in the last part of our considerations.

### 2.2.5 Additional Parts

In the last sections we discussed in some detail, how the SCET Lagrangian (2.45) for collinear quarks is obtained. A completely analogous Lagrangian with corresponding fields follows with $n \leftrightarrow \bar{n}$ for the anti-collinear fields.

In addition to those Lagrangians, a part for heavy quark field can be relevant, which is the Lagrangian of heavy-quark effective theory (HQET). We do not discuss the latter here. A discussion can be found for example in [18].

Moreover, the gluon only part of the Lagrangian is relevant. The parts involving only $A_c^\mu$ or $A_{us}^\mu$, respectively, take the same form as in QCD. There is, however, also a part with interactions between both types of fields. More detail can be found in [17], for example, where it was also argued how for appropriate gauge choices these interactions can be removed at leading power in $\lambda$ through field redefinitions, analogously to what we illustrated for the Lagrangian in eqn. (2.45).

Above, we discussed ultra-soft modes scaling as $(\lambda^2, \lambda^2, \lambda^2)$ and have seen how their contribution to leading power in $\lambda$ can be described by ultra-soft Wilson lines only. The consideration of soft modes scaling as $(\lambda, \lambda, \lambda)$ is technically more complicated, as their virtuality is not suppressed w.r.t. (anti)-collinear modes and their contribution in the $n$ ($\bar{n}$) direction is parametrically larger than the one of the (anti)-collinear mode. From considering the propagators, one finds, that the corresponding fields scale as $A^\mu \sim (\lambda, \lambda, \lambda)$ and $\psi \sim \lambda^{3/2}$. Therefore, the contribution of the quark is again suppressed w.r.t. the (anti)-collinear sector, while the gluon component in the $n$ ($\bar{n}$) direction leads to the dominant contribution in the interaction with the (anti)-collinear modes. The separation of those modes can be performed from moving from SCET-I to SCET-II in a second matching step. We will not give any details to this steps here. However, as might be guessed from the scaling of the fields, for our purpose the result will be similar to the one of the ultra-soft modes: At leading order in $\lambda$ the soft modes decouple from the (anti)-collinear modes after a field redefinition using a soft Wilson line. The later is similar to eqn. (2.43), but with a soft gluon field in place of the ultra-soft gluon field and the multipole expanded argument being $x_- + x_\perp$ in place of $x_-$. In the kinematic situation and order in $\lambda$ we are interested in, the ultra-soft modes will not contribute, while depending on the regularization chosen in the calculation of the objects of our interest, the soft modes can in principle contribute. Therefore, we will only include the soft modes in our following discussion. The final result will be equivalent to the one of the two step matching. The contribution of the ultra-soft modes can be recovered and confirmed to be trivial from our result by replacing the soft Wilson lines in there with ultra-soft Wilson lines.

Note again that the SCET Lagrangian differs in its form from Lagrangians we usually find in EFTs. Even at leading order in the expansion parameter it involves through the Wilson lines terms with arbitrarily many fields and is non-local. However, the Wilson lines and the Lagrangian have a very specific form and define a meaningful effective theory. In the following section, we will encounter applications of SCET.

# 3 Factorization Theorems

Let us now derive the factorization theorems for the specific class of processes we are interested in. This is the production of color-neutral final states at hadron colliders with high invariant mass at small transverse momentum. Through the invariant mass $M$ and the transverse momentum $p_T$ of the final states, the process involves two separate scales. To achieve factorization, we have to separate their contribution into different pieces. This can be achieved through SCET. In this section, we will discuss the relevant steps and several important properties of the objects we will encounter. To this end, we first consider Drell-Yan and then Higgs production as examples. Among others, this will introduce the transverse parton distribution functions (TPDFs) which are also known as beam functions. The determination of those will be the main part of our work in this thesis.

## 3.1 Drell-Yan Production

The first process we consider is the production of a lepton pair via a virtual photon of momentum $q^\mu$ and invariant mass $M^2 = q^2$ from the collision of the two hadrons $N_1$ and $N_2$ of momentum $p$ and $\bar{p}$, respectively. We discussed the kinematics in subsection 2.2.1 and discussed details about the construction of the relevant effective field theory in the last section. In this section, we will match the matrix element of the Drell-Yan process onto SCET operators. To this end we will follow [5, 19].

### 3.1.1 Matching onto SCET

The matrix element is given by

$$d\sigma = \frac{4\pi\alpha^2}{3q^2 s}\frac{d^4q}{(2\pi)^4}\int \mathrm{d}^4x\, e^{-iq\cdot x}(-g_{\mu\nu})\langle N_1(p)N_2(\bar{p})|J^{\mu\dagger}(x)J^\nu(0)|N_1(p)N_2(\bar{p})\rangle\,, \tag{3.1}$$

with the electro-magnetic current $J^\mu = \sum_q e_q \bar{q}\gamma^\mu q$ containing the rescaled electromagnetic charges $|e_q| = \frac{1}{3}, \frac{2}{3}$. For brevity we suppress the sum over quark flavors and the electric charge $e_q$ in intermediate steps. The first step is to match the current on an effective current operator in SCET as [20]

$$\begin{aligned} J^\mu(x) &\to C_V(-q^2 - i\epsilon, \mu)\big(\bar{\xi}_{\bar{c}} W_{\bar{c}}\big)(x)\gamma_\perp^\mu\big(W_c^\dagger \xi_c\big)(x) \\ &= C_V(-q^2 - i\epsilon, \mu)\big(\bar{\xi}_{\bar{c}}^{(0)} W_{\bar{c}}^{(0)}\big)(x)\gamma_\perp^\mu\big(S_{\bar{n}} S_n\big)(x)\big(W_c^{(0)\dagger}\xi_c^{(0)}\big)(x)\,, \end{aligned} \tag{3.2}$$

which holds at leading order in $\lambda$. In the second line we used the redefined (anti)-collinear fields as introduced in eqn. (2.44) such that the three parts do not interact with each other. Instead of ultra-soft Wilson lines we use soft Wilson lines here. The discussion of ultra-soft modes works in the same way. As we will see from (3.13) they do not contribute, however. $n$ and $\bar{n}$ are LC vectors in the direction of the momenta $p$ and $\bar{p}$ of the colliding hadrons.

$C_V(-q^2 - i\epsilon, \mu)$ is the Wilson coefficient. It contains the physics of the hard modes and depends on the invariant mass $q^2$ of the Drell-Yan pair, which is, up to corrections suppressed by $\lambda^2$, equal to the product of the large light-cone momentum-components $\bar{n} \cdot p_1$ and $n \cdot p_2$ of the two quark fields. As $C_V$ has a branch cut along the positive real $q^2$ axis, the $i\epsilon$ prescription has been introduced. It is listed in section C.4. Due to the non-locality of the SCET Lagrangian, the matching relation (3.2) has an extended form w.r.t. eqn. (2.10). It includes Wilson lines summing up arbitrarily many integrals over gluon fields. Nevertheless, the relation still separates a hard matching coefficient from the effective fields which live at a lower scale.

Let us now express the differential cross section (3.1) with the help of eqn. (3.2) in terms of SCET operators. In the intermediate steps we will suppress the factor $|C_V(-q^2 - i\epsilon, \mu)|^2$. Using a Fierz transformation, we rewrite

$$J^{\dagger\mu}(x)J^{\nu}(0) \to \bar{\chi}_c^d(x)\gamma_\perp^\mu \chi_c^e(0) \;\; \bar{\chi}_{\bar{c}}^f(0)\gamma_\perp^\nu \chi_{\bar{c}}^g(x) \;\; \big(S_n^\dagger S_{\bar{n}}\big)^{dg}_{(x)}\big(S_{\bar{n}}^\dagger S_n\big)^{fe}_{(0)}\,, \tag{3.3}$$

where $\chi_{c,\bar{c}}(x) = \big(W_{c,\bar{c}}^{(0)\dagger}\xi_{c,\bar{c}}^{(0)}\big)(x)$ and $d, e, f, g$ are $SU(N_c)$ color indices with sum over them understood. As the collinear, anti-collinear and soft fields do not interact with each other at leading order in $\lambda$ and the initial state can be decomposed in corresponding states as $\langle N_1(p)N_2(\bar{p})| = \langle N_1(p)| \otimes \langle N_2(\bar{p})| \otimes \langle 0|$, this allows us to break the considered matrix element (3.1) into a product of three matrix elements, which each contains fields of only one region.

Averaging over the color of the external particles we can write $\langle N_1|\bar{\chi}_c^d(x)\gamma_\perp^\mu\chi_c^e(0)|N_1\rangle = \frac{1}{N_c}\delta_{de}\langle N_1|\bar{\chi}_c^h(x)\gamma_\perp^\mu\chi_c^h(0)|N_1\rangle$ and similar for the second factor. Absorbing the two Kronecker $\delta$s into the soft factor to close the sum over color indices to a trace in color space, each individual factor contains closed color sums and we can drop the explicit color indices in the following. We then obtain

$$\begin{aligned}\langle N_1(p)N_2(\bar{p})|J^{\mu\dagger}(x)J^\nu(0)|N_1(p)N_2(\bar{p})\rangle \to \frac{1}{N_c^2}\langle N_1(p)|\bar{\chi}_c(x)\gamma_\perp^\mu\chi_c(0)|N_1(p)\rangle \\ \times \langle N_2(\bar{p})|\bar{\chi}_{\bar{c}}(0)\gamma_\perp^\nu\chi_{\bar{c}}(x)|N_2(p)\rangle\langle 0|\mathrm{Tr}\,\bar{T}\big[S_n^\dagger(x)S_{\bar{n}}(x)\big]T\big[S_{\bar{n}}^\dagger(0)S_n(0)\big]|0\rangle\,.\end{aligned} \tag{3.4}$$

In the correlator of the soft Wilson lines, we explicitly wrote the time and anti-time ordering operators $T$ and $\bar{T}$. They arise, as we silently used the Keldysh formalism [21, 22] as explained in Appendix C of [19], to obtain a path integral formulation of the matrix elements above, which have not been time ordered. This formalism specifies the ordering of the fields. While the path and anti-path ordering in the (anti)-collinear functions implies the appropriate ordering there, for the soft function we had to specify it explicitly. In our later perturbative calculation of matrix elements related to $\langle N_1(p)|\bar{\chi}_c(x)\gamma_\perp^\mu\chi_c(0)|N_1(p)\rangle$ and the anti-collinear variant of this, the formalism implies that the fields left of $\gamma_\perp^\mu$ interact with each other in terms of normal Feynman rules, while the fields to the right interact with complex conjugated Feynman rules. The only interaction between fields of different sides is in terms of a cut propagator. We will see in chapter 4 in more detail, how the matrix elements are perturbatively determined.

We continue the reformulation of eqn. (3.4) by using $\not{n}\chi_c$, $\not{\bar{n}}\chi_{\bar{c}} = 0$ and $\chi_c = \frac{\not{n}\not{\bar{n}}}{4}\chi_c\,, \chi_{\bar{c}} = \frac{\not{\bar{n}}\not{n}}{4}\chi_{\bar{c}}$, which follow directly from the corresponding properties of $\xi_c$ and $\xi_{\bar{c}}$, such that we can simplify the Dirac structure in the contraction of eqn. (3.4) with $g_{\mu\nu}$. Effectively this replaces there $\gamma_\perp^\mu$ by $\frac{\not{\bar{n}}}{2}$ and $\gamma_\perp^\nu$ by $\frac{\not{n}}{2}$.

Finally, we multipole expand the individual fields and drop contributions subleading in $\lambda$. As $x$ in eqn. (3.1) is the Fourier conjugate of the momentum $q \sim M(1,1,\lambda)$ of the photon, it scales as $x \sim M^{-1}(1,1,\lambda^{-1})$. The derivatives on the various fields scale like the momenta of the corresponding region, which have been given in table 2.1. In $x\cdot\partial = x_+\partial_- + x_-\partial_+ + x_\perp\partial_\perp$ the leading contribution is $\lambda^0$ and we can drop the subleading components. The latter are $x_-$ for the collinear fields, $x_+$ for the anti-collinear fields, $x_+$ and $x_-$ for the soft fields, and all components of $x$ for the ultra-soft fields. In the multipole expansion only the remaining components have to be kept. We thus receive

$$\begin{aligned}\langle N_1(p)N_2(\bar{p})| - J^{\mu\dagger}(x)J_\mu(0)|N_1(p)N_2(\bar{p})\rangle &\to \frac{1}{N_c^2}\langle N_1(p)|\bar{\chi}_c(x_+ + x_\perp)\frac{\not{\bar{n}}}{2}\chi_c(0)|N_1(p)\rangle \\ &\times \langle N_2(\bar{p})|\bar{\chi}_{\bar{c}}(0)\frac{\not{n}}{2}\chi_{\bar{c}}(x_- + x_\perp)|N_2(p)\rangle\langle 0|\mathrm{Tr}\,\bar{T}\big[S_n^\dagger(x_\perp)S_{\bar{n}}(x_\perp)\big]T\big[S_{\bar{n}}^\dagger(0)S_n(0)\big]|0\rangle\,,\end{aligned} \tag{3.5}$$

where the matrix element with the collinear fields is the only one depending on $x_+$, while the one with the anti-collinear fields is the only one depending on $x_-$.

### 3.1.2 Transverse PDFs

The large momentum components of the photon can be related to those of the hadrons as $\bar{n}\cdot q = z_1\,\bar{n}\cdot p$ and $n\cdot q = z_2\,n\cdot\bar{p}$ with the fractions $z_1 = \sqrt{\tau}e^y, z_2 = \sqrt{\tau}e^{-y} \in [0,1]$ expressed in terms of the rapidity $y$ and the longitudinal energy fraction $\tau = \frac{q^2+q_T^2}{s}$ of the photon with respect to the center of mass energy $s$. Moreover, we split the integral over $x$ in eqn. (3.1) into light-cone and transverse components as $\int\mathrm{d}^4x\, e^{-iq\cdot x} = \frac{1}{2}\int\mathrm{d}^2x_\perp\, e^{-iq_\perp\cdot x_\perp}\int\mathrm{d}t_1\, e^{-iz_1t_1\bar{n}\cdot p}\int\mathrm{d}t_2\, e^{-iz_2t_2n\cdot\bar{p}}$, where we substituted $n\cdot x = t_1$ and $\bar{n}\cdot x = t_2$.

In terms of the quantities appearing in the differential cross section, we then define

$$\mathcal{B}_{q/N_1}(z_1, x_T^2, \mu) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}t_1\; e^{-iz_1t_1\bar{n}\cdot p}\big\langle N_1(p)\big|\bar{\chi}_c(t_1\bar{n} + x_\perp)\frac{\not{\bar{n}}}{2}\chi_c(0)\big|N_1(p)\big\rangle\,, \tag{3.6}$$

$$\bar{\mathcal{B}}_{\bar{q}/N_2}(z_2, x_T^2, \mu) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}t_2\; e^{-iz_2t_2n\cdot\bar{p}}\big\langle N_2(\bar{p})\big|\bar{\chi}_{\bar{c}}(0)\frac{\not{n}}{2}\chi_{\bar{c}}(t_2n + x_\perp)\big|N_2(\bar{p})\big\rangle\,, \tag{3.7}$$

$$\mathcal{S}(x_T^2, \mu) = \frac{1}{N_c}\langle 0|\mathrm{Tr}\,\bar{T}\big[S_n^\dagger(x_\perp)S_{\bar{n}}(x_\perp)\big]T\big[S_{\bar{n}}^\dagger(0)S_n(0)\big]|0\rangle\,. \tag{3.8}$$

Keep in mind that the effective quark field $\chi = W^{(0)\dagger}\xi^0$ contains a Wilson line summing up all eikonal interactions with (anti)-collinear gluons in the direction of large energy flow. As indicated by the arguments $x_T^2 = -x_\perp^2$, the functions above do not depend on the orientation of $x_\perp^\mu$ in the transverse plane. $\mathcal{B}_{q/N_1}(z_1, x_T^2, \mu)$ is the naive transverse parton distribution function (nTPDF) for the collinear region. It describes the distribution of collinear quarks with longitudinal momentum fraction $z_1$ and transverse displacement $x_T^2 = -x_\perp^2$ inside a hadron $N_1$ of momentum $p$. Analogously, the nTPDF $\bar{\mathcal{B}}_{\bar{q}/N_2}$ describes the distribution of anti-collinear anti-quarks with transverse displacement $x_T^2 = -x_\perp^2$ and longitudinal momentum fraction $z_2$ in a hadron $N_2$ of momentum $\bar{p}$. Note that the (anti)-quark's transverse displacement $x_T$ is related to it's transverse momentum by Fourier transformation. By exchanging $N_1(p)$ with $N_2(\bar{p})$ (which implies the exchange of $n$ and $\bar{n}$) and $c$ with $\bar{c}$ in eqns. (3.6) and (3.7), we obtain the quark nTPDF of the anti-collinear region and the anti-quark nTPDF of the collinear region, respectively. The (n)TPDFs will be the focus of this thesis.

In terms of them and $\mathcal{S}$, we can write the differential cross section as

$$\begin{aligned} d\sigma = \frac{4\pi\alpha^2}{3N_c q^2 s}|C_V(-q^2,\mu)|^2 \frac{dq^4}{2(2\pi)^2}\int \mathrm{d}^2x_\perp\, e^{-iq_\perp\cdot x_\perp}\mathcal{S}(x_T^2,\mu) \\ \times \sum_q e_q^2 \left[\mathcal{B}_{q/N_1}(z_1,x_T^2,\mu)\bar{\mathcal{B}}_{\bar{q}/N_2}(z_2,x_T^2,\mu) + (q \leftrightarrow \bar{q})\right] , \end{aligned} \tag{3.9}$$

which holds up to corrections in $\lambda^2 = \frac{q_T^2}{q^2}$. By the sum over quark flavors and the terms with exchanged quark and anti-quark indicated in the last line, we restored the formerly suppressed sum over quarks and corresponding anti-quarks.

Fixing the invariant mass of the photon to $M^2$, its rapidity to $y$ and the absolute value of its transverse momentum to $q_T$, but performing the integral over the angle in the transverse plane, we obtain the triple differential cross section $d^3\sigma/(dM^2 dq_T^2 dy)$ as the RHS of eqn. (3.9) with $q^2$ replaced by $M^2$ and $d^4q$ replaced by $2\pi$:

$$\begin{aligned} \frac{d^3\sigma}{dM^2 dq_T^2 dy} = \frac{4\pi\alpha^2}{3N_c M^2 s}|C_V(-M^2,\mu)|^2 \frac{1}{4\pi}\int \mathrm{d}^2x_\perp\, e^{-iq_\perp\cdot x_\perp}\mathcal{S}(x_T^2,\mu) \\ \times \sum_q e_q^2 \left[\mathcal{B}_{q/N_1}(z_1,x_T^2,\mu)\bar{\mathcal{B}}_{\bar{q}/N_2}(z_2,x_T^2,\mu) + (q \leftrightarrow \bar{q})\right] . \end{aligned} \tag{3.10}$$

Up to a caveat, we will resolve around eqn. (3.16) this is the final all order factorization formula for the differential cross section of the Drell-Yan process at small transverse momentum. It appears to separate the hard scale $M^2$ from the smaller scale $x_T^{-2}$.

### 3.1.3 Generalization

The factorization formula (3.10) can be generalized to the production of other color neutral final states with invariant mass $M^2$ and transverse momentum $q_T$. As discussed in [5], for the production of a $Z$ boson we replace the sum over quark flavors in eqn. (3.10) by

$$\sum_q e_q^2 \to \sum_q \frac{|g_L^q|^2 + |g_R^q|^2}{2} = \sum_q \frac{(1-2|e_q|\sin^2\theta_W)^2 + 4e_q^2\sin^4\theta_w}{8\sin^2\theta_W\cos^2\theta_W}\,, \tag{3.11}$$

and for the production of a $W^-$ boson, we replace it by

$$\sum_q e_q^2 \to \sum_{q,q'} \frac{|g_L^{qq'}|^2}{2} = \sum_{q,q'} \frac{|V_{q,q'}|^2}{4\sin^2\theta_W}\,, \tag{3.12}$$

with the weak mixing angle $\theta_W$ and the CKM matrix elements $V_{q,q'}$. In addition to that any other process with a color neutral final state of high invariant mass and small transverse momentum can be factorized in a form analogous to eqn. (3.10). In many cases we will however need gluon nTPDFs in addition to the quark nTPDFs defined above. Those we will introduce in section 3.2.

Let us also comment on the function $\mathcal{S}(x_T^2,\mu)$. It contains the physics of the modes scaling as $\sim(\lambda,\lambda,\lambda)$. If we would instead have considered the ultra-soft modes scaling as $\sim(\lambda^2,\lambda^2,\lambda^2)$, we would have found in place of $\mathcal{S}(x_T^2,\mu)$ by the same considerations the correlator of ultra-soft Wilson lines

$$\mathcal{Y}(0) = \frac{1}{N_c}\langle 0|\mathrm{Tr}\,\bar{T}\big[Y_n^\dagger(0)Y_{\bar{n}}(0)\big]T\big[Y_{\bar{n}}^\dagger(0)Y_n(0)\big]|0\rangle = \langle 0|0\rangle = 1\,, \tag{3.13}$$

where also the $x_\perp$ dependence is subleading and has been dropped. As this factor is equal to 1 it is irrelevant for our considerations and can be dropped.

In our calculation also the soft factor $\mathcal{S}(x_T^2,\mu)$ will collapse to the factor 1 [6]. However, this is specific about the regularization, which we will choose. In a general case, this function can contribute. We will see below, that the individual functions $\mathcal{S}$, $\mathcal{B}$ and $\bar{\mathcal{B}}$ are not well defined scheme independently, but only their product is. The physical reason is that the soft, collinear and anti-collinear modes cannot be separated unambiguously. All of them have the same virtuality $\lambda^2$. To distinguish them one can consider their different rapidities. However, in intermediate rapidity regions there is some ambiguity how to distribute the contribution to the various modes which implies an ambiguity of the individual functions. Details about the approach used by us follow in section 4.1.

### 3.1.4 PDFs and Matching Kernels

Note that the multipole expansion performed for eqn. (3.5) to identify the function $\mathcal{B}$ and $\bar{\mathcal{B}}$ depends on the kinematics. In a kinematic situation, where we would not be sensitive to the small transverse momentum, but have e.g. $x\sim(1,1,1)$, also the $x_\perp$ dependence would be subleading. Then we would have found

$$\phi_{q/N_1}(z_1,\mu) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t_1\, e^{-iz_1 t_1 \bar{n}\cdot p} \big\langle N_1(p) \big| \bar{\chi}(t_1\bar{n}) \frac{\bar{\not{n}}}{2} \chi(0) \big| N_1(p) \big\rangle \tag{3.14}$$

for the collinear region and a corresponding expression with appropriate replacements for the anti-collinear region as well a standard factorization formula in terms of those functions. The functions $\phi$ are the normal (anti)-collinear PDFs and the equation above provides their operator definition.

Comparing eqn. (3.14) with the one for $\mathcal{B}$, it becomes obvious that nTPDFs are a generalization of normal PDFs, for which the dependence on the transverse components must be kept. This motivates the name given to them. Physically the need to keep the transverse dependence of these functions is the sensitivity of the differential cross section to this scale due to the fact that the initial state radiation has to balance the transverse momentum of the vector boson.

If $x_T^{-2}$ is in the perturbative region, i.e. $x_T^{-2} \gg \Lambda_{\mathrm{QCD}}$, the transverse PDF can be related to the normal PDF by an OPE as [5, 23, 24]

$$\mathcal{B}_{i/N}(z, x_T^2, \mu) = \sum_{j=q,\bar{q},g} \int_\rho^1 \frac{d\rho}{\rho} \mathcal{I}_{i/j}(\rho, x_T^2, \mu) \phi_{j/N}(z/\rho, \mu) + \mathcal{O}(\Lambda^2 x_T^2)\,, \tag{3.15}$$

which defines the matching kernel $\mathcal{I}_{i/j}$ for the partons $i$ and $j$. Summing over all partons $j$ and Mellin convoluting the matching kernels with the PDFs thus yields the nTPDF. Hence, the matching kernel contains all the perturbative physics of the nTPDF at the scale $x_T^2$ and it will allow us to express the differential cross section in terms of perturbative quantities and normal PDFs. It can be extracted from this equation and a perturbative calculation of the partonic (T)PDFs, where we replace the hadron $N$ by a parton $k$. This highly non-trivial perturbative calculation will be performed in the following chapters.

### 3.1.5 Refactorization

A main complication in this calculation is that dimensional regularization alone is not sufficient to regulate all singularities. This can be considered a consequence of decomposing the full theory into individual regions, where each individual region can be more singular than the combination of all regions. To obtain well defined integrals, we use an additional analytic regulator $\alpha$ in the way proposed in [10]. We will discuss the exact form of this regulator and a corresponding regularization mass scale $v$ in sections 4.1 and 4.9, here we will only state the main implications. With the analytic regulator $\alpha$ and the dimensional regulator $\epsilon$ all appearing integrals are well defined. The soft function collapses to a trivial factor 1. However, the result of $\mathcal{B}$ and $\bar{\mathcal{B}}$ will contain poles not only in the dimensional regulator $\epsilon$ but also in the analytic regulator $\alpha$. Hence, the individual nTPDFs are not well defined independently from the analytic regulator, but only the product of a collinear and corresponding anti-collinear nTPDF is. In this product, all poles in $\alpha$ cancel and the regulator can be dropped. The corresponding scale $v$ occurs as $\log \frac{v\bar{n}\cdot p}{\mu^2}$ in the collinear and as $\log \frac{v}{\bar{n}\cdot\bar{p}}$ in the anti-collinear region. In the product of the nTPDFs of the two regions, consistently also the dependence on this unphysical scale cancels, once the regulator is dropped. However, at the same time a dependence on the hard scale $q^2$ ($\sim \bar{n}\cdot p\, \bar{n}\cdot\bar{p} z_1^{-1} z_2^{-1}$) is introduced. This effect is called collinear anomaly [5]. More details about these points are provided in section 8.1, where we also discuss that the $q^2$ dependence can be refactorized by [5]

$$\left[\mathcal{S}(x_T^2,\mu,v)\mathcal{B}_{q/N_1}(z_1,x_T^2,\mu,v)\bar{\mathcal{B}}_{\bar{q}/N_2}(z_2,x_T^2,\mu,v)\right]_{q^2} \overset{\alpha=0}{=} \left(\frac{x_T^2 q^2}{4e^{-2\gamma_e}}\right)^{-F_{q\bar{q}}(x_T^2,\mu)} B_{q/N_1}(z_1,x_T^2,\mu)B_{\bar{q}/N_2}(z_2,x_T^2,\mu)\,. \tag{3.16}$$

This defines the process independent TPDFs $B$ and the anomaly exponent $F$. All three are properly independent of the large scale $q^2$, the analytic regulator $\alpha$ and the corresponding scale $v$. Moreover, there is no distinction between collinear and anti-collinear versions for the $B$ functions despite its different arguments. In eqn. (3.16) we included the in our case trivial factor $\mathcal{S}$ to allow easier contact to related considerations in different frameworks, e.g. [25, 26]. The dependence on the hard scale is controlled to all orders in perturbation theory by the first factor on the RHS of eqn. (3.16).

Only with this refactorization, the factorization of the differential cross section is complete. It then takes the form

$$\begin{aligned}\frac{d^3\sigma}{dM^2 dq_T^2 dy} &= \frac{\alpha^2}{3N_c M^2 s}\left|C_V(-M^2,\mu)\right|^2 \int d^2x_\perp\, e^{-iq_\perp\cdot x_\perp}\left(\frac{x_T^2 M^2}{4e^{-2\gamma_e}}\right)^{-F_{q\bar{q}}(x_T^2,\mu)} \\ &\times \sum_{i,j=q,\bar{q},g}\sum_q e_q^2\left[B_{q/N_1}(z_1,x_T^2,\mu)B_{\bar{q}/N_2}(z_2,x_T^2,\mu)+(q\leftrightarrow\bar{q})\right]+\mathcal{O}\left(\frac{q_T^2}{M^2}\right).\end{aligned} \tag{3.17}$$

This is the proper, all order factorization formula for the differential cross section of the Drell-Yan process at small transverse momentum which achieves the consistent separation of the hard scale $M^2$ from the collinear scale $x_T^{-2}$. While the hard part $|C_V|^2$ is process dependent, the anomaly exponent $F$ and the TPDFs $B$ are process independent.

The factorization formula takes the usual form

$$d\sigma \sim \mathcal{H}\otimes\mathcal{S}\otimes\mathcal{C}\otimes\bar{\mathcal{C}}\,, \tag{3.18}$$

where in our case the soft part $\mathcal{S}$, which cannot be determined independently of the (anti)-collinear parts $\mathcal{C}$ and $\bar{\mathcal{C}}$, has been absorbed in the latter ones. To obtain the factorization formula above, we have followed SCET based arguments presented in [5]. However, factorization formulae and approaches to resummation for the Drell-Yan and other processes have been discussed in full QCD already long before SCET was developed. Very notably is the formalism by Collins, Soper and Sterman (CSS) presented among others in [7], where the differential Drell-Yan cross section was factorized in the sense of eqn. (3.18).

We will discuss the relation to this approach in section 3.1.9. One great benefit of the systematic derivation using SCET is that within the framework each quantity appearing in the differential cross section is defined explicitly or can be directly obtained from such quantities.

The factorization formula above holds for all values of $x_T^2$. At perturbative values of this scale, we are able to perform an additional step and relate the TPDFs to normal PDFs. To this end, in analogy to eqn. (3.15) we introduce the matching kernels $I_{i/j}$ as

$$B_{i/N}(z, x_T^2, \mu) = \sum_{j=q,\bar{q},g} I_{i/j}(z, x_T^2, \mu) \otimes \phi_{j/N}(z, \mu) + \mathcal{O}(\Lambda^2 x_T^2)\,, \tag{3.19}$$

where here and in the following, we use the shorthand notation (B.34) for the Mellin convolution. Just as the (T)PDFs, the matching kernels are process independent.

We then can rephrase the differential cross section (3.17) as

$$\begin{aligned} \frac{d^3\sigma}{dM^2 dq_T^2 dy} = \frac{\alpha^2}{3N_c M^2 s} \sum_{i,j=q,\bar{q},g} \sum_q e_q^2 \int_{z_1}^1 \frac{d\xi_1}{\xi_1} \int_{z_2}^1 \frac{d\xi_2}{\xi_2} \\ \times \left[ C_{q\bar{q}\leftarrow ij}(\xi_1, \xi_2, q_T^2, M^2, \mu) \phi_{i/N_1}(z_1/\xi_1, \mu) \phi_{j/N_2}(z_2/\xi_2, \mu) + (q \leftrightarrow \bar{q}) \right] \,, \end{aligned} \tag{3.20}$$

which holds up to power corrections in $q_T^2/M^2$ and $x_T^2 \Lambda_{\mathrm{QCD}}^2$ with the perturbative function

$$\begin{aligned} C_{q\bar{q}\leftarrow ij}(\xi_1, \xi_2, q_T^2, M^2, \mu) = \left| C_V(-M^2, \mu) \right|^2 \int d^2 x_\perp \, e^{-iq_\perp \cdot x_\perp} \left( \frac{x_T^2 M^2}{4e^{-2\gamma_e}} \right)^{-F_{q\bar{q}}(x_T^2, \mu)} \\ \times I_{q\leftarrow i}(\xi_1, x_T^2, \mu) I_{\bar{q}\leftarrow j}(\xi_2, x_T^2, \mu)\,. \end{aligned} \tag{3.21}$$

Now the great benefit of the systematic framework we are applying becomes fully apparent. Each function in $C$ is exactly defined in the framework. Moreover, it only depends on its native mass scale and the renormalization scale. Hence, each of them can be consistently obtained by a straightforward perturbative calculation of the relevant operators and appropriate subsequent steps.

In the chapters to follow, we will make use of the operator definitions of $\mathcal{B}$ and $\bar{\mathcal{B}}$ to extract among others the process independent matching kernels $I$ and the anomaly coefficient $F$ to NNLO in perturbation theory.

### 3.1.6 Renormalization

Rushing to the final factorization formula above, we have not yet mentioned several other relevant points. Despite the poles in $\alpha$, the functions $\mathcal{B}$, $\bar{\mathcal{B}}$ also have poles in $\epsilon$ regulating the UV and IR divergences. The poles in $\epsilon$ will also appear in their product, such that also $B$ and $F$ have such poles and require renormalization. We perform the renormalization in the $\overline{\mathrm{MS}}$-scheme. The renormalization of the coupling constant is inherited from full QCD and described in section 4.8. In addition, we also need operator renormalization in the spirit of eqn. (2.13), since we are working in an effective field theory. For the (T)PDFs, the anomaly exponent and the Wilson coefficient they take the form

$$\phi_{i/N}^{(b)}(z) = \sum_k Z_{i/k}^{\phi}(z,\mu) \otimes \phi_{k/N}^{(r)}(z,\mu)\,, \tag{3.22}$$

$$B_{i/N}^{(b)}(z,x_T^2) = Z_i^B(x_T^2,\mu) B_{i/N}^{(r)}(z,x_T^2,\mu)\,, \tag{3.23}$$

$$F_{q\bar{q}}^{(b)}(x_T^2) = Z_q^F(\mu) + F_{q\bar{q}}^{(r)}(x_T^2,\mu)\,, \tag{3.24}$$

$$C_V^{(b)}(-q^2) = Z^{C_V}(-q^2,\mu) C_V^{(r)}(-q^2,\mu)\,, \tag{3.25}$$

such that all three factors on the RHS of eqn. (3.16) are renormalized in a multiplicative way. Eqns. (3.19, 3.22, 3.23) imply the renormalization equation for $I_{i/k}$. We introduced the labels $(b)$ and $(r)$ to distinguish bare and renormalized expressions. Throughout this thesis, we work in the $\overline{\text{MS}}$-scheme. There, the renormalization constants $Z$ are pure poles in $\epsilon$ and the renormalized functions are free of UV poles. During most of our earlier and following discussion we suppress the indices $(b)$ and $(r)$.

### 3.1.7 RGE Equations

As usual, the renormalization introduces a dependence on the renormalization scale $\mu$. Physical quantities as the cross section (3.10) cannot depend on this scale, such that the derivative w.r.t. this scale has to vanish. This implies that the RGE equations of the various factors in the cross section are not independent, but the sum of their kernels has to vanish. The RGE equation of the Wilson coefficient, in terms of which the hard function is given, reads

$$\frac{d}{d\log\mu} C_V(-q^2,\mu) = \left[\Gamma^q(\alpha_s)\log\frac{-q^2}{\mu^2} + 2\gamma^q(\alpha_s)\right] C_V(-q^2,\mu)\,, \tag{3.26}$$

where $\Gamma^q$ is the cusp anomalous dimension in the fundamental representation and $\gamma^q$ is the quark anomalous dimension, which both are listed in Appendix C.1.

RGE invariance of the cross section and eqns. (3.26, 3.16) then imply the RGE equations

$$\frac{d}{d\log\mu} F_{q\bar{q}}(x_T^2,\mu) = 2\,\Gamma^q(\alpha_s)\,, \tag{3.27}$$

$$\frac{d}{d\log\mu} B_{q/N}(z,x_\perp^2,\mu) = \left[\Gamma^q(\alpha_s)\log\frac{x_T^2\mu^2}{4e^{-2\gamma_e}} - 2\gamma^q(\alpha_s)\right] B_{q/N}(z,x_\perp^2,\mu)\,, \tag{3.28}$$

such that the RGE dependence of the RHS of eqn. (3.16) exactly compensates the one of hard function. For the logarithm appearing in the last equation, we will introduce the symbol

$$L_\perp = \log\frac{x_T^2\mu^2}{4e^{-2\gamma_e}}\,. \tag{3.29}$$

Eqn. (3.28) and the standard DGLAP equation

$$\frac{d}{d\log\mu}\phi_{j/N}(z,\mu) = \sum_k 2P_{jk}(z,\mu)\otimes\phi_{k/N}(z,\mu) \tag{3.30}$$

imply the RGE equations

$$\begin{aligned}\frac{d}{d\log\mu}I_{i/j}(z,x_T^2,\mu) =& \Big[\Gamma^q(\alpha_s)L_\perp - 2\gamma^q(\alpha_s)\Big] I_{i/j}(z,x_T^2,\mu)\\ &- \sum_k 2I_{i/k}(z,x_T^2,\mu)\otimes P_{kj}(z,\mu)\,,\end{aligned} \tag{3.31}$$

for $i\in\{q,\bar{q}\}$ and with the well known DGLAP splitting kernels which are listed in Appendix C.3. Also the renormalization factors in eqns. (3.23-3.22) obey RGE equations which exactly compensate the renormalization scale dependence of the corresponding renormalized functions, for example

$$\frac{d}{d\log\mu}Z_q^F(x_T^2,\mu) = -\,2\,\Gamma^q(\alpha_s)\,, \tag{3.32}$$

$$\frac{d}{d\log\mu}Z_q^B(x_T^2,\mu) = -\,\left[\Gamma^q(\alpha_s)L_\perp - 2\gamma^q(\alpha_s)\right] Z_q^B(x_T^2,\mu)\,. \tag{3.33}$$

From these equations and the $\overline{\mathrm{MS}}$ condition, the renormalization factors are implied order by order in terms of the corresponding anomalous dimensions and the QCD $\beta$ function.

### 3.1.8 Resummation

Each individual function in $C$ only depends on its native mass scale and the renormalization scale. Hence, they can be consistently determined in perturbation theory, choosing the renormalization scale for each factor such that the logarithms of scale ratios are small. In a subsequent step, one solves the RGE equations (3.26, 3.27, 3.31) to evolve the functions to a common matching scale. This step resums all large logarithms. These steps are illustrated in figure 3.1. The solutions to the RGE equations are given in the Appendix of [5] for example.

As for all functions but $C_V$ the natural scale is $x_T^2$ and furthermore the RGE equation for the matching kernel $I$ is by the presence of the Mellin convolution more difficult than the other RGE equations, one usually chooses the matching scale such that the logarithms of $x_T^2/\mu_m$ are small. The large logarithms '$L$' are then those containing $q^2/\mu_m^2$ and $x_T^2 q^2$. While the latter are already in an exponentiated form controlled by $F$, the former are obtained by solving the RGE equation (3.26).

We then will obtain a result of the form

$$C = c(\alpha_s)\exp\left[Lg_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \alpha_s^2 g_4(\alpha_s L) + \dots\right], \tag{3.34}$$

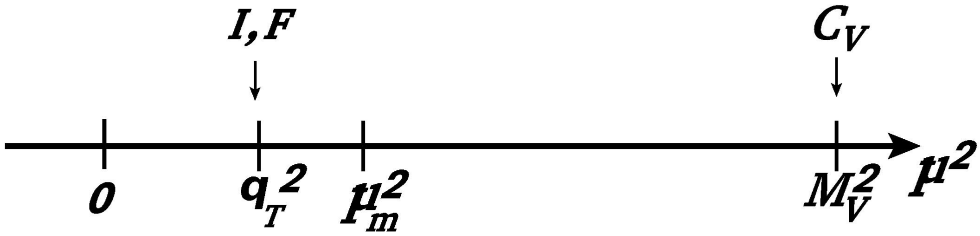



Figure 3.1: Each function is evaluated at its native scale and then evolved to a common matching scale.

where $c$ is a function independent of the large logarithm $L$ and the functions $g_i$ with $g_i(0) = 0$ in the exponent contain all dependence on $L$. Obviously on expanding the exponent, the term with $g_1$ will give rise to the leading logarithms $(\alpha_s L^2)^n$. To obtain the (next-to)$^k$-leading-logarithmic (N$^k$LL) contributions $\alpha_s^k(\alpha_s L^2)^n$, all functions $g_i$ with $i \leq k+1$ must be known. In this sense, $g_i$ resums the N$^{i-1}$LL.

Provided each ingredient is determined to sufficient order, one can from this equation in principle obtain (next-to)$^k$-leading-logarithmic (N$^k$LL) precision for any $k \in \mathbb{N}$. To obtain e.g. N$^3$LL precision, the function $C_V$ and $I$ have to be known to $\alpha_s^2$, while $F$, $P_{kj}$ and $\gamma^q$ are needed to $\alpha_s^3$. Moreover, $\Gamma^q$ and $\beta$ are needed to $\alpha_s^4$.

With the calculation of $I$ to $\alpha_s^2$ in this thesis, we therefore determine an important, process independent part for N$^3$LL resummation.

The QCD $\beta$-function has been determined to $\alpha_s^4$ in [27], $P_{kj}$ has been extracted to $\alpha_s^3$ in [28, 29] and $\gamma^q$ has been obtained to $\alpha_s^3$ in [30]. Despite single vector boson production, there exist a couple of other processes, for which $C_V$ is known to sufficient order, for example those considered in [31].

The other ingredients are not yet known to the order sufficient for N$^3$LL resummation. So far only the terms of them sufficient for N$^2$LL resummation have been determined. We will determine $F$ to N$^2$LO. This has been achieved already in [5]. $\Gamma^q$ has been determined to N$^3$LO in [28].

### 3.1.9 Comparison to CSS

Having covered all points relevant for the factorization and resummation of Drell-Yan like processes in the framework of [5], we can compare this framework to the standard formalism of transverse momentum resummation according to CSS in [7]. Also there, at leading power the differential cross section can be factorized in the form (3.20). The perturbative function $C_{q\bar{q}\leftarrow ij}$ is now represented differently by

$$C_{q\bar{q}\leftarrow ij}(\xi_1, \xi_2, q_T^2, M^2, \mu) = \int d^2x_\perp\, e^{-iq_\perp \cdot x_\perp} C_{qi}(z_1, \alpha_s(\mu_x)) C_{\bar{q}j}(z_2, \alpha_s(\mu_x)) \times \exp\left\{ -\int_{\mu_x}^{M^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[ \log \frac{M^2}{\bar{\mu}^2}\, A(\alpha_s(\bar{\mu})) + B(\alpha_s(\bar{\mu})) \right] \right\}, \quad (3.35)$$

with $\mu_x = 2e^{2\gamma_e} x_T^{-1}$ . Expressing those functions by the functions appearing in eqn. (3.21), one finds [5]

$$
\begin{aligned}
C_{qi}(z_1, \alpha_s(\mu_x)) &= \left|C_V(-\mu_x^2, \mu_x)\right| I_{q/i}(z_1, x_T^2, \mu_x) \,, \\
C_{\bar{q}j}(z_2, \alpha_s(\mu_x)) &= \left|C_V(-\mu_x^2, \mu_x)\right| I_{\bar{q}/j}(z_2, x_T^2, \mu_x) \,, \\
A(\alpha_s(\bar{\mu})) &= \Gamma^q(\alpha_s) - \bar{\mu}^2 \frac{dF_{q\bar{q}}(\bar{x}_T^2, \bar{\mu})}{d\bar{\mu}^2} \,, \\
B(\alpha_s(\bar{\mu})) &= 2\gamma^q(\alpha_s) + F_{q\bar{q}}(\bar{x}_T^2, \bar{\mu}) - \bar{\mu}^2 \frac{d \log |C_V(-\bar{\mu}^2, \bar{\mu})|^2}{d\bar{\mu}^2} \,,
\end{aligned}
\tag{3.36}
$$

with the anomalous dimensions which appeared in the RGE equations. The scale choice $\mu_x = 2e^{2\gamma_e} x_T^{-1}$ in the CSS approach eliminates all logarithms $L_\perp$, but $\alpha_s(\mu_x)$ is doomed to hit the Landau pole of the running coupling as the expression involves an integral over $x_T$. Therefore a regularization prescription is needed. Practically, the integral is cut off at large values $x_T$. [5] also discuss different scale choices and the rewriting of their formula to $q_T$ space. Thus, in their framework, we can avoid hitting the Landau pole without using a hard cut-off.

### 3.1.10 Some Remarks

While the dependence of the combined collinear and anti-collinear regions on the hard scale revealed in eqn. (3.16), might be perceived an anomaly, this dependence is actually required by consistency of the theory. As discussed above, due to the RGE independence of the physical cross section (3.10), the RG evolution of the functions appearing in there have to compensate. The RGE equation of the Wilson coefficient (3.27) however contains a logarithm of the hard scale $q^2$, which has to be compensated by the RGE equations of the remaining factors. Thus, these factors have in fact to generate a dependence on the hard scale. Seen from this perspective the arising of the hard scale $q^2$ in the product $\mathcal{B}\bar{\mathcal{B}}$ is not surprising, but required by consistency.

We have discussed above that the RGEs for $I$ and $F$, which were obtained by corresponding considerations, can be used for obtaining the final resummed result. In addition to that, for our calculation of the TPDFs, they can serve as powerful checks for the consistency of our results. This and other checks will be discussed in section 8.5.

Despite Drell-Yan-like processes discussed in this section, we will consider general processes where a heavy, color neutral final state with small transverse momentum is created. To this end, we will generalize our framework, discussed in this section to also hold for gluon initiated processes. As an important example, we will discuss Higgs production in the next section. Among others, this will lead us to the introduction of the gluon TPDFs.

## 3.2 Higgs Production

Many aspects in the derivation of the factorized cross section for Higgs production at small transverse momentum and the consideration of its ingredients are analogous to Drell-Yan production. However, there are two relevant differences. An additional step integrating out the top quark precedes the matching procedure and the Higgs production is then mediated through gluon-gluon fusion. This will lead us to the introduction of gluon TPDFs. In our considerations we will follow [6, 32, 33].

### 3.2.1 Matching onto the Effective Theory

The kinematic considerations correspond exactly to those of the Drell-Yan production in section 2.2.1. Therefore, also the same modes will be relevant in our consideration.

The first step integrates out the energetic modes about and above the top quark mass $m_t$ in the spirit of section 2.1. This introduces the Wilson coefficient $C_t$ which is given in Appendix C.4. With this coefficient the effective Lagrangian describing the Higgs production via gluon-gluon fusion is

$$\mathcal{L}_{\text{eff}}(x) = C_t(m_t^2, \mu)\frac{\alpha_s(\mu)}{12\pi}\frac{H(x)}{v_h}G^A_{\mu\rho}(x)G^{A,\mu\rho}(x)\,, \tag{3.37}$$

with the Standard Model Higgs field $H$, the vacuum expectation value $v_h$ and the gluon field tensor $G^A_{\mu\rho} = \partial_\mu A^A_\rho - \partial_\rho A^A_\mu + g_s f^{ABC} A^B_\mu A^C_\rho$, where $A$, $B$, $C$ are adjoint $SU(N_c)$ indices with a sum over them understood. The differential cross section for Higgs production is then given by

$$\begin{aligned} d\sigma =& \frac{1}{2s}\left(\frac{\alpha_s(\mu)}{12\pi v_h}\right)^2 C_t^2(m_t^2,\mu)\frac{d^4q}{(2\pi)^4}\int d^4x e^{-iq\cdot x} \\ &\times \langle N_1(p)N_2(\bar{p})|G^A_{\mu\rho}G^{A,\mu\rho}(x)G^B_{\nu\sigma}G^{B,\nu\sigma}(0)|N_1(p)N_2(\bar{p})\rangle\,, \end{aligned} \tag{3.38}$$

with the two incoming hadrons $N_1$ and $N_2$ of momentum $p$ and $\bar{p}$, respectively, giving rise to their combined invariant mass $s = (p+\bar{p})^2$. The next step is the transition to SCET. All steps are considering the leading order in the expansion parameter $\lambda = q_T^2/q^2$ The gluon operator above is then matched on a SCET operator build from gauge invariant building blocks as introduced in eqn. (2.49),

$$G^A_{\mu\rho}G^{A,\mu\rho}(x) \to -2q^2 C_S(-q^2 - i\epsilon, \mu) g^\perp_{\mu\rho} \mathcal{A}^{A,\mu}_c (S^\dagger_n S_{\bar{n}})^{AB} \mathcal{A}^{B,\rho}_{\bar{c}}(x)\,. \tag{3.39}$$

The hard matching coefficient, derived together with this matching equation in [34], is listed in Appendix C.4. We suppress the regulating $i\epsilon$ prescription in its argument in the following. The transverse metric tensor, which describes the tensor structure of the hard function, can be written in terms of the two hadron momenta $p$ and $\bar{p}$ as

$$g^{\perp}_{\mu\rho} = g_{\mu\rho} - \frac{p^{\mu}\bar{p}^{\nu} + p^{\nu}\bar{p}^{\mu}}{p \cdot \bar{p}} \,. \tag{3.40}$$

The soft Wilson lines in eqn. (3.39) are introduced through the decoupling transformation of soft fields from the (anti)-collinear fields. Note that above they are in the adjoint representation.

Since at leading order in $\lambda$ the soft, collinear and anti-collinear fields do not interact with each other, we can factorize this operator and the initial state $\langle N_1(p)N_2(\bar{p})| = \langle N_1(p)| \otimes \langle N_2(\bar{p})| \otimes \langle 0|$ in a collinear, anti-collinear and soft piece, such that the matrix element in eqn. (3.38) factorizes correspondingly as

$$\begin{aligned} d\sigma = {}& \frac{1}{2s}\left(\frac{\alpha_s(\mu)}{12\pi v_h}\right)^2 C_t^2(m_t^2,\mu)\big|C_S(-q^2,\mu)\big|^2 g^{\perp}_{\mu\rho} g^{\perp}_{\nu\sigma} \frac{d^4q}{(2\pi)^4} 4(q^2)^2 \int d^4x e^{-iq\cdot x} \\ & \times \langle N_1(p)|\mathcal{A}_c^{A,\mu}(x)\mathcal{A}_c^{D,\nu}(0)|N_1(p)\rangle \langle N_2(\bar{p})|\mathcal{A}_{\bar{c}}^{B,\rho}(x)\mathcal{A}_{\bar{c}}^{C,\sigma}(0)|N_2(\bar{p})\rangle \\ & \times \langle 0|\bar{T}[S_n^{\dagger}(x)S_{\bar{n}}(x)]^{AB}\, T[S_{\bar{n}}^{\dagger}(0)S_n(0)]^{CD}|0\rangle \,. \end{aligned} \tag{3.41}$$

The time- and anti-time-ordering symbols arise as in the Drell-Yan case through the Keldysh formalism.

Averaging over the color of the external particles we can write $\langle N_1|\mathcal{A}^{A,\mu}(x)\mathcal{A}^{D,\nu}(0)|N_1\rangle = \frac{1}{N_c^2-1}\delta^{AD}\langle N_1|\mathcal{A}^{E,\mu}(x)\mathcal{A}^{E,\nu}(0)|N_1\rangle$ and similar for the second factor. We then absorb the two Kronecker $\delta$s into the soft factor to close the sum over adjoint color indices, such that there is no color correlation left between the different factors.

### 3.2.2 Transverse PDFs

Finally, we multipole expand each factor in eqn. (3.41), introduce the energy fractions $z_1$, $z_2$ and split the integral over $x$ into light-cone and transverse components. All these steps have been explained in detail around eqn. (3.5). After their application to eqn. (3.41), we can therein identify the soft function

$$\mathcal{S}(x_T^2,\mu) = \frac{1}{N_c^2-1}\langle 0|\mathrm{Tr}\,\bar{T}\big[S_n^{\dagger}(x_\perp)S_{\bar{n}}(x_\perp)\big]^{AB} T\big[S_{\bar{n}}^{\dagger}(0)S_n(0)\big]^{BA}|0\rangle \,, \tag{3.42}$$

the collinear gluon nTPDF

$$\mathcal{B}^{\mu\nu}_{g/N_1}(z,x_\perp,\mu) = -\frac{z\bar{n}\cdot p}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-izt\bar{n}\cdot p}\,\langle N(p)|\mathcal{A}_{c\perp}^{A,\mu}(t\bar{n}+x_\perp)\mathcal{A}_{c\perp}^{A,\nu}(0)|N(p)\rangle \,, \tag{3.43}$$

and the anti-collinear gluon nTPDF $\bar{\mathcal{B}}^{\rho\sigma}_{g/N_2}$, which is given by a corresponding expression with $(c, N_1, p \sim n) \leftrightarrow (\bar{c}, N_2, \bar{p} \sim \bar{n})$. We introduced the label $\perp$ on the effective gluon field as a reminder that only the transverse components are considered. We also note that in contrast to the Drell-Yan case, the gluon nTPDFs are Lorentz tensors. This tensor structure will be analyzed in section 3.2.5.

In terms of these quantities, the differential cross section for Higgs production can be written as

$$\frac{d^2\sigma}{dq_T^2 dy} = \sigma_0(\mu) C_t^2(m_t^2, \mu) \big| C_S(-q^2, \mu) \big|^2 g^{\perp}_{\mu\rho} g^{\perp}_{\nu\sigma} \frac{1}{2\pi} \int \mathrm{d}^2 x_\perp \, e^{-iq_\perp \cdot x_\perp} \\ \times \mathcal{S}(x_T^2, \mu) \mathcal{B}^{\mu\nu}_{g/N_1}(z_1, x_\perp, \mu) \bar{\mathcal{B}}^{\rho\sigma}_{g/N_2}(z_2, x_\perp, \mu) \,, \tag{3.44}$$

which holds up to corrections in $\lambda^2 = \frac{q_T^2}{m_H^2}$. $y$ is the rapidity of the Higgs boson, $z_1$, $z_2 = \sqrt{\tau} e^{\pm y}$, with $\tau = (m_H^2 + q_T^2)/s$, $q^2$ is fixed at $m_H^2$, we expanded the overall factor factor $\frac{m_H^2}{\tau s} = 1 + \mathcal{O}(\lambda^2)$ and introduced the Born-level cross section

$$\sigma_0(\mu) = \frac{m_H^2 \alpha_s^2(\mu)}{72\pi (N_c^2 - 1) s v_h^2} \,. \tag{3.45}$$

Note that not only the nTPDFs, but also the hard function, which appears in front of the integral in eqn. (3.44), is a tensor. Instead of considering the production of a single scalar particle, we can also consider the production of a more complicated color neutral final state via gluon-gluon fusion. Then, we find a factorization formula corresponding to (3.44), but with the hard tensor replaced with the one relevant in that case. However, the gluon nTPDFs are still as defined in eqn. (3.43). Hence, they are the generic functions relevant for the gluon induced production of a color neutral final state with large invariant mass and small transverse momentum. To allow for the generalization to other tensor structures of the hard function, we will not restrict ourselves to a specific contraction of their indices in the rest of this section.

Most aspects about the differential cross section (3.44) are completely analogous to what we already observed for the Drell-Yan production. Considering ultra-soft gluons, we would find a ultra-soft Wilson-line correlator evaluated at 0, which simplifies to a trivial factor 1. Moreover, with the regulator which we will choose in our calculation, the contribution of soft gluons, $\mathcal{S}$, simplifies to a trivial factor 1.

This regulator is required for the perturbative calculation of the individual nTPDFs which then contain poles in that regulator. These poles and the associated scale cancel in the product of the collinear and corresponding anti-collinear nTPDF and at the same time reveal a hidden $m_H^2$ dependence. By the same arguments and in a similar form as for the Drell-Yan case, the latter can be refactorized as

$$[\mathcal{S}(x_T^2, \mu, v) \mathcal{B}^{\mu\nu}_{g/N_1}(z_1, x_\perp, \mu, v) \bar{\mathcal{B}}^{\rho\sigma}_{g/N_2}(z_2, x_\perp, \mu, v)]_{m_H^2} \overset{\alpha=0}{=} \\ \left( \frac{x_T^2 m_H^2}{4e^{-2\gamma_e}} \right)^{-F_{gg}(x_T^2, \mu)} B^{\mu\nu}_{g/N_1}(z_1, x_\perp, \mu) B^{\rho\sigma}_{g/N_2}(z_2, x_\perp, \mu) \,, \tag{3.46}$$

which introduces the process independent TPDFs $B$ and the anomaly coefficient $F_{gg}$. They are independent of the additional regulator, the associated scale and the hard scale $m_H$, but depend only on the transverse displacement $x_\perp$, the renormalization scale $\mu$ and, in case of the TPDF, on the energy fraction $z_1$ or $z_2$. As pointed out earlier, we can consider the production of a final state with general spin structure. For this reason, the equation holds for uncontracted Lorentz indices.

### 3.2.3 Renormalization and RGE

The analytic regularization, whose effects we just discussed, is used in addition and independently to dimensional regularization. Hence the functions we just introduced, still have poles in the corresponding regulator $\epsilon$, which are removed by renormalization. The renormalization of the coupling constant is inherited from full QCD and in addition all factors in eqn. (3.46) receive a multiplicative operator renormalization, which in full analogy to eqns. (3.23, 3.24) for $F$ and $B$ take the form

$$B^{(b)\mu\nu}_{g/N}(z, x_\perp) = Z^B_g(x_T^2, \mu) B^{(r)\mu\nu}_{g/N}(z, x_\perp, \mu)\,, \tag{3.47}$$

$$F^{(b)}_{gg}(x_T^2) = Z^F_g(\mu) + F^{(r)}_{gg}(x_T^2, \mu)\,. \tag{3.48}$$

Together with the renormalization of the coupling, this removes all UV poles. However, at the same time a dependence on the renormalization scale $\mu$ is introduced. The functional dependence of the various objects on this scale is described by their RGE equations. For the Wilson coefficients, they are give by

$$\frac{d}{d\log\mu} C_t(m_t^2, \mu^2) = \gamma^t(\alpha_s) C_t(m_t^2, \mu^2)\,, \quad \text{with} \quad \gamma^t = \alpha_s^2 \frac{d}{d\alpha_s} \frac{\beta(\alpha_s)}{\alpha_s^2} \tag{3.49}$$

and

$$\frac{d}{d\log\mu} C_S(-m_h^2 - i\epsilon, \mu) = \left[\Gamma^g(\alpha_s) \log\frac{-m_H^2 - i\epsilon}{\mu^2} + \gamma^S(\alpha_s)\right] C_S(-m_H^2 - i\epsilon, \mu) \tag{3.50}$$

where $\beta$ is the QCD $\beta$-function, $\Gamma^g$ is the cusp anomalous dimension in the adjoint representation and $\gamma^s = 2\gamma^g - \gamma^t - \beta/\alpha_s$ with the gluon anomalous dimension $\gamma^g$. The latter, $\Gamma^g$ and $\beta$ are listed in Appendix C.4. The other anomalous dimensions can be obtained from these. $\gamma^S$ can also be found in [34]. Note in particular that the full hard function, consisting of the factors in front of the integral on the RHS of eqn. (3.44), fulfills a RGE like the hard function $|C_V|^2$ of Drell-Yan production, but with $q^2$, $\Gamma^q$ and $\gamma^q$ replaced by $m_H^2$, $\Gamma^g$ and $\gamma^g$. Then it is easy to see that these equations together with the RGE invariance of the total cross section imply the RGE equations for $B^{\mu\nu}$ and $F_{gg}$ as

$$\frac{d}{d\log\mu} F_{gg}(x_T^2, \mu) = 2\,\Gamma^g(\alpha_s)\,, \tag{3.51}$$

$$\frac{d}{d\log\mu} B^{\mu\nu}_{g/N}(z, x_\perp, \mu) = \left[\Gamma^g(\alpha_s) \log\frac{x_T^2\mu^2}{4e^{-2\gamma_e}} - 2\gamma^g(\alpha_s)\right] B^{\mu\nu}_{g/N}(z, x_\perp, \mu)\,. \tag{3.52}$$

Just as in the quark case, for $x_T^{-2} \gg \Lambda_{\mathrm{QCD}}$, the transverse PDF can be related to the normal PDF by an OPE,

$$B^{\mu\nu}_{g/N}(z, x_\perp, \mu) = \sum_{j=q,\bar{q},g} I^{\mu\nu}_{g/j}(z, x_\perp, \mu) \otimes \phi_{j/N}(z, \mu) + \mathcal{O}(\Lambda^2 x_T^2)\,, \tag{3.53}$$

which defines the matching kernel $I^{\mu\nu}_{g/j}$ containing all perturbative information of the TPDF at the scale $x_\perp$. With a corresponding relation including $\mathcal{I}^{\mu\nu}_{g/j}$, also $\mathcal{B}^{\mu\nu}_{g/N}$ can be matched to the normal PDFs.

The RGE eqn. for $B^{\mu\nu}$ and the DGLAP eqns. (3.30) imply the RGE of $I^{\mu\nu}$ as

$$\begin{aligned}\frac{d}{d\log\mu} I^{\mu\nu}_{g/j}(z, x_\perp, \mu) = &\left[\Gamma^g(\alpha_s) L_\perp - 2\gamma^g(\alpha_s)\right] I^{\mu\nu}_{g/j}(z, x_\perp, \mu) \\ &- \sum_k 2 I^{\mu\nu}_{g/k}(z, x_\perp, \mu) \otimes P_{kj}(z, \mu)\,.\end{aligned} \tag{3.54}$$

### 3.2.4 Factorized Differential Cross Section

In terms of the expressions introduced here, we can give the final factorization formula for the differential cross section for Higgs production at small transverse momentum as

$$\begin{aligned}\frac{d^2\sigma}{dq_T^2 dy} = &\,\sigma_0(\mu) C_t^2(m_t^2, \mu) \left|C_S(-m_H^2, \mu)\right|^2 g^\perp_{\mu\rho} g^\perp_{\nu\sigma} \frac{1}{2\pi} \int \mathrm{d}^2 x_\perp\, e^{-iq_\perp \cdot x_\perp} \\ &\times \left(\frac{x_T^2 m_H^2}{4e^{-2\gamma_e}}\right)^{-F_{gg}(x_T^2, \mu)} B^{\mu\nu}_{g/N_1}(z_1, x_\perp, \mu) B^{\rho\sigma}_{g/N_2}(z_2, x_\perp, \mu)\,,\end{aligned} \tag{3.55}$$

which holds up to corrections in $\lambda^2 = \frac{q_T^2}{m_H^2}$. Following the SCET based arguments of [6,33], we thus received a factorization of the standard form (3.18) with clearly defined individual quantities. If $q_T^2 \gg \Lambda_{\mathrm{QCD}}$ we can use the matching kernels, to relate the TPDFs to normal PDFs and then rewrite

$$\begin{aligned}\frac{d^2\sigma}{dq_T^2 dy} = &\,\sigma_0(\mu) \sum_{i,j=q,\bar{q},g} \int_{z_1}^1 \frac{d\xi_1}{\xi_1} \int_{z_2}^1 \frac{d\xi_2}{\xi_2} C_{gg\leftarrow ij}(\xi_1, \xi_2, q_T^2, m_H^2, m_t, \mu) \\ &\times \phi_{i/N_1}(z_1/\xi_1, \mu)\, \phi_{j/N_2}(z_2/\xi_2, \mu)\,,\end{aligned} \tag{3.56}$$

which holds up to power corrections in $q_T^2/M^2$ and $\Lambda^2_{\mathrm{QCD}}/q_T^2$ with the perturbative function

$$\begin{aligned}C_{gg\leftarrow ij}(\xi_1, \xi_2, q_T^2, m_H^2, m_t, \mu) = &\,C_t^2(m_t^2, \mu) \left|C_S(-m_H^2, \mu)\right|^2 g^\perp_{\mu\rho} g^\perp_{\nu\sigma} \frac{1}{2\pi} \int d^2 x_\perp\, e^{-iq_\perp \cdot x_\perp} \\ &\times \left(\frac{x_T^2 m_H^2}{4e^{-2\gamma_e}}\right)^{-F_{gg}(x_T^2, \mu)} I^{\mu\nu}_{g/i}(\xi_1, x_\perp, \mu) I^{\rho\sigma}_{g/j}(\xi_2, x_\perp, \mu)\,,\end{aligned} \tag{3.57}$$

This factorization formula consistently separates all physical mass scales, such that each function depends only on a single such scale. As explained in more detail in section 3.1.8, each function can be evaluated consistently choosing $\mu$ at its native scale and is then evolved to a common matching scale by solving the RGE equations. In the last step the large logarithms are resummed to all orders, with a logarithmic accuracy depending on the perturbative order the various ingredients are known. The functions $I_{g/k}$ we are going to determine up to NNLO in the main part of this thesis, are an ingredient needed to obtain N$^3$LL accuracy.

### 3.2.5 Additional Aspects

So far, we have not provided the operator definition of the normal gluon PDF. It is found, when considering, in analogy to the consideration leading to eqn. (3.14), the factorization for a final state without the restriction to small transverse momentum. Then the multipole expansion for the various sectors would change and particularly we could drop the transverse component of $x$ in the collinear sector to find the normal gluon PDF

$$\phi_{g/N}(z,\mu) = -g^{\perp}_{\mu\nu}\frac{z\bar{n}\cdot p}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}t\, e^{-izt\bar{n}\cdot p}\,\langle N(p)|\mathcal{A}^{A,\mu}_{c\perp}(t\bar{n})\mathcal{A}^{A,\nu}_{c\perp}(0)|N(p)\rangle\,, \tag{3.58}$$

and correspondingly for the anti-collinear sector. The transverse metric tensor has been given in eqn. (3.40). Due to the multipole expansion the PDF does not depend on $x_\perp$. Thus, it has, in contrast to the TPDF, only a trivial Lorentz structure.

The Lorentz structure of the TPDF, on the other hand, can be decomposed in its two independent parts

$$B^{\mu\nu}_{g/N}(z,x_\perp,\mu) = \frac{g^{\mu\nu}_\perp}{d-2}\,B_{g/N}(z,x_T^2,\mu) + \left(\frac{g^{\mu\nu}_\perp}{d-2}+\frac{x^\mu_\perp x^\nu_\perp}{x_T^2}\right)B'_{g/i}(z,x_T^2,\mu)\,, \tag{3.59}$$

where just as in eqn. (3.43) only transverse indices are considered and we defined the two scalar functions $B$ and $B'$, which are given by the contractions of $B^{\mu\nu}$ with the projectors

$$P^{\mu,\nu}_{g_\perp} = g^{\mu\nu}_\perp \qquad \text{and} \qquad P^{\mu,\nu}_{x} = \frac{d-2}{d-3}\left[\frac{g^{\mu\nu}_\perp}{d-2}+\frac{x^\mu_\perp x^\nu_\perp}{x_T^2}\right]\,, \tag{3.60}$$

respectively. In our later perturbative calculation of the partonic TPDFs, $B_{g/k}$ and $B_{q/k}$ will lead to integrals and results of the same type, while some integrals and the final result for $B'_{g/k}$ have a different form. This motivates the reuse of the symbol $B$.

In the same way as we decompose $B^{\mu\nu}$, we can also decompose $I^{\mu\nu}$, $\mathcal{B}^{\mu\nu}$ and $\mathcal{I}^{\mu\nu}$, giving rise to $I$, $I'$; $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{I}$, $\mathcal{I}'$, respectively.

The contraction of the two matching kernels with the metric tensor in eqn. (3.57) at

$d = 4$ results in

$$2\, g^{\perp}_{\mu\rho}\, g^{\perp}_{\nu\sigma}\, I^{\mu\nu}_{g/i}(\xi_1, x_\perp, \mu) I^{\rho\sigma}_{g/j}(\xi_2, x_\perp, \mu) = \\ I_{g/i}(\xi_1, x_T^2, \mu) I_{g/j}(\xi_2, x_T^2, \mu) + I'_{g/i}(\xi_1, x_T^2, \mu) I'_{g/j}(\xi_2, x_T^2, \mu)\,. \tag{3.61}$$

Hence, in case of the production of a single scalar particle, the two tensor structures do not mix. In our later calculation, we will find $I' = \mathcal{O}(\alpha_s)$, while $I$ starts at order $\alpha_s^0$. To obtain a given accuracy of the differential Higgs cross section, among others eqn. (3.61) has to be know to corresponding order. Then in this case, it will be sufficient to determine $I'$ to one perturbative order less then $I$. For a final state with general spin, this does not hold however.

Summing up, in this chapter we have seen how SCET can be applied to find the factorization theorems for the production of color neutral final states with high invariant mass and small transverse momentum. We explicitly discussed the two examples of Drell-Yan and Higgs production. We moreover pointed out generalizations to both considerations. Out analysis has lead us directly to the operator definitions for the nTPDFs. We have observed how an anomalous dependence on the hard scale in the product of the collinear and anti-collinear nTPDF arises and how it can be properly refactorized introducing the process independent TPDFs and anomaly coefficients. We also specified a relation which related the TPDFs to normal PDFs and provided explicit factorization formulas for various processes. We moreover have discussed how the RGE equations can be used to resum all large logarithms.

In this context, we have observed that one process independent ingredient required to obtain N$^3$LL precision are the NNLO matching kernels. This motivates the relevance of our corresponding NNLO calculation which we present in the following chapters.

# 4 Perturbative Calculation

In the last chapters, we have introduced a framework which allows to separate physics of different scales and to consistently resum large logarithms. This has lead us to operator definitions of the nTPDFs. We have also seen that they are a generalization of normal PDFs and how both of them are related through the matching kernels at perturbative values of $x_T^{-2}$. In particular, we have argued how a calculation of the parton-to-parton nTPDFs and PDFs allows the extraction of these process independent matching kernels. As the determination of latter and the anomaly coefficients up to NNLO in perturbation theory is the aim of our work, we will now discuss how the calculation of the underlying parton-to-parton nTPDFs is performed. With simple adjustments, the normal PDFs can be determined in a corresponding way, which will be pointed out after eqn. (4.6).

In this section we will focus on general aspects and provide details to the explicit calculations of the non-trivial contributions in the chapters to follow. For explicitness, we will focus on the collinear quark nTPDF. However, analogous statements hold for anti-quark, gluon and anti-collinear nTPDFs.

Let us rewrite eqn. (3.6) slightly by replacing the hadron $N_1(p)$ by a parton $b(p)$ and by introducing the identity in form of a sum over all collinear, partonic, intermediate states $X$ integrated over all momenta. Suppressing the collinear labels here and in the following, we then can write

$$\mathcal{B}_{q/b}(z, x_T^2, \mu, v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-izt\bar{n}\cdot p} \sum_X \frac{\bar{\not{n}}_{\alpha\beta}}{2} \langle b(p)|\bar{\chi}_\alpha(t\bar{n}+x_\perp)|X\rangle\langle X|\chi_\beta(0)|b(p)\rangle\,. \quad (4.1)$$

## 4.1 Analytic Regularization

The sum over intermediate states includes for each parton $i$ in $X$ an integral over the corresponding momentum $l_i$, which is constrained to be physical, i.e. $l_i^2 = 0$ and $l_i^0 \geq 0$. This integration contains in addition to the usual dimensional regulator $\epsilon = \frac{4-d}{2}$, the analytic regulator and an associated scale $v$. More precisely following [10], the integral of each parton $i$ in $X$ is of the form

$$\int \frac{\mathrm{d}^d l_i}{(2\pi)^{d-1}} \delta^+(l_i^2) \left(\frac{v}{n\cdot l_i}\right)^\alpha . \quad (4.2)$$

This or an alternative additional regulator is necessary, since dimensional regularization is not sufficient to regulate all singularities in the integrals over the light-cone momenta. They contain singularities originating from regions of high absolute values of rapidity.

The regulator regulated and effectively discriminates regions of high positive and negative rapidity. Note that this regulator appears with the same LC component for the collinear and anti-collinear region. Without the regulator, those regions did directly correspond to each other via $p \sim n \leftrightarrow \bar{p} \sim \bar{n}$.

Closely related to this way of additional regularization is the one suggested in [25]. Yet another way of regularization has been suggested in [26]. From a calculational aspect, we expect that the regulator in (4.2) is the most feasible one, as it is scaleless and only appears on LC propagators of the momenta of partons in $X$.

## 4.2 Matrix Elements and Feynman Rules

Let us come back to another aspect of eqn. (4.1) which demands additional explanation. That is the effective quark field and its conjugate $\bar{\chi}(x) = \bar{\xi}(x)W(x)$, introduced in eqn. (2.48) with a Wilson line of the form of eqn. (2.32). The later resums an arbitrary number of collinear gluons and is necessary to conserve local gauge invariance of the non-local operator (4.1). These issues have been explained in more detail in section 2.2. The Wilson line also contains a path-ordering. The corresponding path starts way back in the $\bar{n}^\mu$-direction and stretches all the way along the LC vector $\bar{n}^\mu$ to the point $x$. The path ordering can be identified as a time ordering and we can move the quark field inside the ordering operator to receive $\langle b(p)|T\bar{\xi}(x)W(x)|X\rangle$ with $x = x_\perp + t\bar{n}$. For the second part, we obtain the Hermitian conjugate of this expression with the time ordering replaced by an anti-time ordering and evaluated at $x = 0$.

Let

$$k = \sum_{i \in X} l_i$$

be the sum of all momenta contained in $X$. Using overall momentum conservation, we then can write the matrix element in eqn. (4.1), as $e^{i(p-k)\cdot(x_\perp + t\bar{n})}|\langle b(p)|T\bar{\xi}(0)W(0)|X\rangle|^2$. This is the square of a time ordered amplitude with a special phase space factor.

The $x_\perp$-dependent part of the phase space factor $e^{i(p-k)\cdot(x_\perp+t\bar{n})}$ is equal to $e^{-ik_\perp \cdot x_\perp}$, since $p_\perp = 0$. For the $t$-dependent part we can perform the $t$-integral in eqn. (4.1) leading to

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-izt\bar{n}\cdot p} e^{it\bar{n}(p-k)} = \delta(\bar{n}\cdot k - (1-z)\bar{n}\cdot p) = \delta(k_- - z_- p_-)\,, \tag{4.3}$$

where in the last step we used the short notation of section 2.2.1 and use from here on

$$z_- = (1-z)\,. \tag{4.4}$$

For the contribution, where $n$ partons are exchanged, we can then identify the matrix element

$$m_{q/b,\{j_i\}_n,\alpha} = \langle \{j_i(l_i)\}_n | T\bar{\xi}_\alpha(0)W(0)|b(p)\rangle \tag{4.5}$$

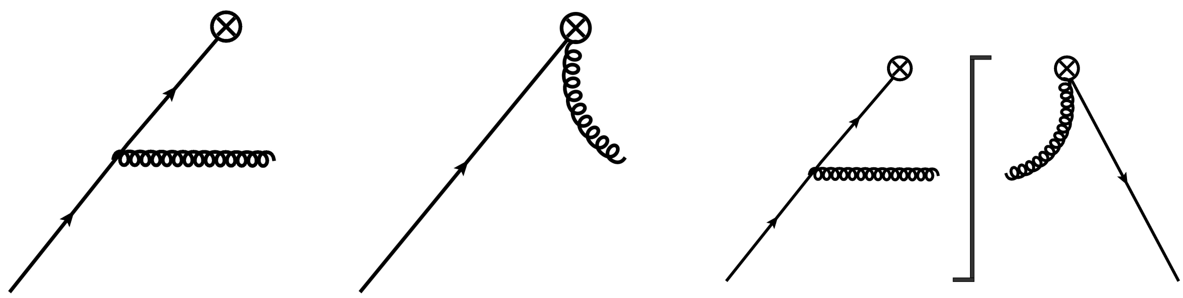


Figure 4.1: (a,b): The two diagrams contributing to $\langle g|T\bar{\chi}_\alpha(0)|q\rangle$. (c): One possible combination.

and in terms of it write the nTPDF as

$$\mathcal{B}_{q/b}(z, x_T^2, \mu, v) = \sum_n \left[\prod_{i=1,\dots,n} \int \frac{\mathrm{d}^d l_i}{(2\pi)^{d-1}} \delta^+(l_i^2) \left(\frac{v}{n \cdot l_i}\right)^\alpha\right] \int \mathrm{d}^d k\, \delta^d(k - \sum_i l_i) \\ \times e^{-ik_\perp \cdot x_\perp}\, \delta(k_- - z_- p_-) \sum_{\{j_i(l_i)\}_n} \frac{\bar{\not{n}}_{\alpha\beta}}{2} |m_{q/b,\{j_i\}_n}|^2_{\alpha\beta}\,. \tag{4.6}$$

The definition and therefore the integral kernel of the usual PDF only differs from this by lacking the $x_\perp$-dependence. It is therefore given by the same expression with the factor $e^{-ik_\perp \cdot x_\perp}$ removed.

The matrix element, we can express in the normal way in terms of Feynman diagrams. For the coupling to the effective quark field $\chi = W^\dagger \xi$, to which in addition to a quark any number of gluons can couple, we introduce the symbol '$\otimes$'. The two diagrams for the example $\langle g(l)|T\bar{\chi}_\alpha(0)|q(p)\rangle$ as well as one of the four combinations relevant for $|m|^2$ are given in Figure 4.1. Since the matrix element involves only fields of the collinear region, we can and will equivalently use QCD Feynman rules, instead of SCET Feynman rules, to evaluate the expression. The ones relevant for gluons coupling to the special vertex $\otimes$, can be straightforwardly obtained by expanding the Wilson line in the coupling constant $g$ and using Wick's theorem. For the coupling of the quark with momentum $k_0$ and $n$ gluons with momentum $k_i$, color $B_i$ and Lorentz index $\beta_i$ to the effective quark field, we obtain

$$\mathrm{FR}_{\bar{\chi}} = g_s^n \sum_\sigma \prod_{j=n}^{1} \frac{\bar{n}^{\beta_{\sigma(j)}}}{\sum_{l=j}^n \bar{n}\cdot k_{\sigma(l)}} T^{B_{\sigma(j)}} \tag{4.7}$$

with a sum over all permutations $\sigma$ of the gluons.

For the effective coupling of $n+1$ gluons with momentum $k_i$, color $B_i$ and Lorentz index $\beta_i$ labeled by $i = 0, \dots, n$ to the effective gluon field $\mathcal{A}^{\mu, D_{n+1}}$, we receive from similar considerations

$$\mathrm{FR}_{\mathcal{A}} = g_s^{n+1} \sum_{\sigma} \left( \prod_{j=1}^{n} f^{D_j D_{j-1} B_{\sigma(j)}} \right) \left( g^{\mu\beta_{\sigma(0)}} - \frac{k^{\mu}_{\sigma(0)} \bar{n}^{\beta_{\sigma(0)}}}{\sum_{n_0=0}^{n} \bar{n}\cdot k_{n_0}} \right) \left( \prod_{m=1}^{n} \frac{-i\bar{n}^{\beta_{\sigma(m)}}}{\sum_{l=m}^{n} \bar{n}\cdot k_{\sigma(l)}} \right) \tag{4.8}$$

with a sum over all permutations $\sigma$ of the gluons and over the color indices appearing twice. Furthermore, $D_0 = B_0$. These Feynman rules lead to LC propagators of all possible combinations of the momenta involved.

Instead of working in a general gauge, we will from here on choose a special gauge, in which the Wilson lines reduce to a trivial factor 1. Then the effective fields are just the normal quark and gluon fields to which a corresponding parton can couple. This is the case for the LC gauge, where $\bar{n} \cdot A = 0$. The LC vector $\bar{n}$ is the projector to the direction of large energy flow $p \sim n$. In this gauge, the gluon propagator becomes

$$\langle 0|TA^{\mu}(x)A^{\nu}(y)|0\rangle = \int \mathrm{d}^d k e^{ik(x-y)} \frac{1}{k^2} D^{\mu\nu}(\bar{n}, k)\,, \tag{4.9}$$

where we defined

$$D^{\mu\nu}(\bar{n}, k) = \Big[ - g^{\mu\nu} + \frac{k^{\mu}\bar{n}^{\nu} + k^{\nu}\bar{n}^{\mu}}{\bar{n}\cdot k} \Big]\,. \tag{4.10}$$

By the denominator $\bar{n} \cdot k$ in eqn. (4.10), the LC gauge will give rise to LC propagators of the same kind as the Wilson lines did. The final result between both variants is the same, but form our point of view the discussion is more transparent in the LC gauge.

For a fixed choice of partons $j_i$, the sum over states $\{j_i(l_i)\}_n$ in eqn. (4.6) corresponds to sums over their spins and colors. Furthermore we average over spin and color of parton $b$. This closes all spin and color sums in the squared matrix element. Therein, the sum over spins of a external gluon field of momentum $l$ gives rise to $D^{\mu\nu}(\bar{n}, l)$, while the spin sum over an external (anti)-quark with momentum $l$ gives rise to $\gamma_{\mu} l^{\mu}$. If the state $b$ is a gluon, the average factor is $\frac{1}{d-2}\frac{1}{N_c^2-1}$, and if it is a (anti)-quark, the average factor is $\frac{1}{2}\frac{1}{N_c}$.

Also the spin and color sums containing the explicit (anti)-quark field $\overset{(-)}{\xi}$ are connected between these two fields. The spinor sum is connected over the factor $\frac{\bar{\not{n}}_{\alpha\beta}}{2}$, which will therefore appear in the corresponding trace of $\gamma$-matrices.

### 4.2.1 Gluon and Anti-Quark Case

As stated in the beginning of this section, in our discussion here, we focused on the collinear quark nTPDF, but corresponding considerations hold for the anti-quark, gluon and anti-collinear nTPDFs.

For the anti-collinear nTPDFs, we simply have to exchange $n \leftrightarrow \bar{n}$ everywhere, but for the terms $(n \cdot l_i)^{-\alpha}$ raised to powers of the analytic regulator. To allow a parallel discussion of both cases in a simple way, we rename $(p, n) \leftrightarrow (\bar{p}, \bar{n})$ in the anti-collinear sector throughout the perturbative calculation. Then their discussion can be done in the same notation with the only difference that the analytic regulators now appear on $\bar{n} \cdot l_i$ in the anti-collinear sector.

For the anti-quark and gluon nTPDFs, we find equations as (4.6), but with the proper replacement for $\frac{\bar{\not{n}}_{\alpha\beta}}{2} |m_{q/b,\{j_i\}_n}|^2_{\alpha\beta}$. For the anti-quark, $m_{q/b,\{j_i(l_i)\}_n,\alpha}$ is replaced by

$$m_{\bar{q}/b,\{j_i\}_n,\alpha} = \langle \{j_i(l_i)\}_n | T \chi_\alpha(0) | b(p) \rangle \,. \tag{4.11}$$

For the gluon, the relevant matrix elements are

$$m^\mu_{g/b,\{j_i\}_n} = \langle \{j_i(l_i)\}_n | T \mathcal{A}^\mu(0) | b(p) \rangle \,, \tag{4.12}$$

and the factor $-z\,\bar{n}\cdot p\,P_{\mu\nu}$ is introduced in eqn. (4.6) in place of $\frac{\bar{\not{n}}_{\alpha\beta}}{2}$, where $P_{\mu\nu}$ is the projector to one of the two Lorentz structures and given in eqn. (3.60) for both cases. These adjustments obviously follow from the difference between the definitions (3.43) and (4.1).

As before, in the LC gauge, the effective quark field $\chi$ reduces to the normal quark field $\xi$ and the effective gluon field $\mathcal{A}$ to the normal gluon field $A$.

To have a short notation for later use, we write the squared matrix elements summed and averaged over spin and color as

$$\begin{aligned}
\overline{|M^{(n)}_{g/b}|^2} &= \frac{1}{d-2}\frac{1}{N_c^2-1} \sum_{\text{color}} \sum_{\text{spin}} \sum_{\{j_i(l_i)\}_n} -z\,\bar{n}\cdot p\, P_{g_\perp,\mu\nu} |m_{g/b,\{j_i\}_n}|^{2\;\mu\nu} \,, \\
\overline{|M^{(n)}_{\bar{q}/b}|^2} &= \frac{1}{2}\frac{1}{N_c} \sum_{\text{color}} \sum_{\text{spin}} \sum_{\{j_i(l_i)\}_n} \frac{\bar{\not{n}}_{\alpha\beta}}{2} |m_{\bar{q}/b,\{j_i\}_n}|^2_{\alpha\beta} \,.
\end{aligned} \tag{4.13}$$

Moreover, $\overline{|M^{(n)}_{q/b}|^2}$ is as the last equation with $|m_{q/b,\{j_i\}_n}|$ and $\overline{|M'^{(n)}_{g/b}|^2}$ is like the first equation but with $P_{x,\mu\nu}$ at the corresponding places. Then the (nT)PDFs can be written in the form

$$\begin{aligned}
\left\{\mathcal{B}_{a/b},\, \phi_{a/b}\right\}(z) = &\sum_n \left[ \prod_{i=1,\dots,n} \int \frac{\mathrm{d}^d l_i}{(2\pi)^{d-1}} \delta^+(l_i^2) \left(\frac{v}{n\cdot l_i}\right)^\alpha \right] \int \mathrm{d}^d k\; \delta^d(k - \sum_i l_i) \\
&\times \left\{ e^{-ik_\perp\cdot x_\perp},\, 1\right\} \delta(k_- - z_- p_-)\; \overline{|M^{(n)}_{a/b}|^2} \,,
\end{aligned} \tag{4.14}$$

where here and below, we suppress the arguments of $\mathcal{B}$ and $\phi$ given by mass scales.

We want to make the following remark in the context of rewriting the nTPDFs in terms of integrals over squared matrix elements. Instead of introducing the explicit intermediate states and rewriting the path-ordering in the Wilson line in terms of a time-ordering, one can also apply the Keldysh formalism to the initial matrix element to obtain the cut diagrams.

We have shortly commented on it after eqn. (3.4). Also there the transition amplitude is split into a $T$ and a $\bar{T}$ ordered part. The fields only interact between these parts via a cut propagator, which takes the same form as the integrals over the states in $X$. Especially when considering the gluon nTPDF in a general gauge, performing the considerations with this formalism might be preferable.

## 4.3 Automatization

Our main task now will be to determine the perturbative expressions of the matrix elements, square each independent contribution, perform the integrals and simplify the result. The LO contribution is for $\alpha_s^0$ and we want to determine the nTPDFs up to NNLO, i.e. up to $\alpha_s^2$. With each order, the complexity of the calculation increases rapidly. We therefore automatized most aspects of the calculation.

In a top level bash-script we specify the external partons and number of loops. It then calls QGraf [35], equipped with appropriate steering files, to generate the corresponding amplitudes. Using among others the 'sed' command, the bash script rewrites these amplitudes in form format and produces several form modules. Then it calls form [36]. Form identifies the spin and color structures, generates the complex conjugated amplitudes and constructs the squared matrix element. Simplifying between each step, it then plugs in the Feynman rules, works out the color- and gamma-algebra, identifies the integrals, uses relations between them, plugs in their result and expands the expressions in the regulators $\epsilon$ and $\alpha$. The whole process just takes a few seconds.

The solutions to the encountered integrals are explicitly provided by us. Finding these solutions is the main difficulty and will be discussed in the next chapters. There are some aspects, which let these integrals differ from usual QCD integrals. The main aspects can be easily observed from eqn. (4.14). They are the presence of the analytic regulator $\alpha$ on the LC components of the external momenta, the $x_\perp$ dependent exponential and the $\delta$-function constraining $k_-$. In addition to that, we also have to deal with the additional propagators, introduced by either the Wilson lines or LC propagators. Particularly the first two aspects impede us from using many of the known, powerful reduction and integration techniques in an advantageous way. For example, the method of integration by parts according to [37] becomes much less useful. Due to the special form of the integration kernel, we could only apply it for a subset of momenta and the useful reduction of the system of equations suffers from the presence of $\alpha$.

For these reasons, we only used elementary techniques as partial fraction decomposition, tensor reduction, relabeling invariance and shifts of internal loop momenta to reduce the number or complexity of the integrals encountered. The integrals are then solved by choosing suitable parametrizations, separating independent parts and performing the remaining integrals over dimensionless variables with Mathematica. In these steps, we often have to deal with Hypergeometric functions. We make use of several of their analytic properties. Among these, also its series expansion in $\alpha$ and $\epsilon$. To obtain it, we use the package HypExp [38].

## 4.4 Leading Order

Let us illustrate the steps, we are performing, on the toy example of the leading order contributions. That is, we consider the tree level contribution to the term $n = 0$ in the sum in eqn. (4.6). Then no momentum $l_i$ appears, such that $k = 0$ and

$$\mathcal{B}^{(0)}_{q/b}(z) = e^0 \frac{1}{p_-} \delta(1-z) \frac{\bar{\not{n}}_{\alpha\beta}}{2} |m^{0L}_{q/b,0}|^2_{\alpha\beta} \,. \tag{4.15}$$

The new label $0L$ on the matrix element specifies the number of internal loops to be zero. The label $^{(0)}$ on the LHS specifies the order of expansion of $\mathcal{B}$ in $\alpha_s/(4\pi)$. In general, we have

$$\mathcal{B} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n \mathcal{B}^{(n)} \,, \tag{4.16}$$

and despite the LO $n = 0$, we will later discuss also the NLO term $n = 1$ and NNLO term with $n = 2$.

If the parton $b$ in eqn. (4.15) is not a quark, the matrix element vanishes. If it is a quark, it can be contracted with the field $\bar{\xi}$. Averaging over spin and color of the incoming quark then leads to

$$\frac{\bar{\not{n}}_{\alpha\beta}}{2} \overline{|m^{0L}_{q/b,0}|^2}_{\alpha\beta} = \frac{1}{2} \mathrm{Tr}\Big(\not{p} \frac{\bar{\not{n}}}{2}\Big) \frac{1}{N_c} \mathrm{Tr}_c\Big(I\Big) = \bar{n} \cdot p \tag{4.17}$$

and we thus obtain

$$\mathcal{B}^{(0)}_{q/b}(z) = \delta_{qb} \delta(1-z) \,. \tag{4.18}$$

For the gluon nTPDF we perform analogous considerations. The matrix element does vanish if the parton $b$ is not a gluon. If it is a gluon, we average over its color and spin to find

$$-z\, \bar{n}\cdot p\, P_{\mu\nu} \overline{|m^{0L}_{g/g,0}|^2}^{\mu\nu} = -z\, \bar{n}\cdot p\, P_{\mu\nu} \frac{1}{d-2} D^{\mu\nu}(p, \bar{n}) \frac{1}{N_c^2 - 1} \delta_{AA} = z\, \bar{n}\cdot p\, P_{\mu\nu} \frac{g^{\perp\,\mu\nu}}{d-2} \,, \tag{4.19}$$

with $g^{\perp\,\mu\nu}$ and $P_{\mu\nu}$ given in eqns. (3.40) and (3.60), respectively. We therefore find for the two tensor structures

$$\mathcal{B}^{(0)}_{g/b}(z) = \delta_{gb} \delta(1-z) \,, \quad \text{and} \tag{4.20}$$

$$\mathcal{B}'^{(0)}_{g/b}(z) = 0 \,. \tag{4.21}$$

Exactly the same results are obtained for the anti-collinear TPDFs.

## 4.5 Virtual Corrections

The next case, we consider, are purely virtual corrections to $\mathcal{B}_{a/b}$. This is, we consider the matrix element $m^{nL}_{a/b,0}$ with $n$ loops. Neither the analytic regulator and its corresponding scale $v$, nor $x_\perp$ are part of these loop integrals. The only mass scale they contain is the external momentum $p$. However, $p^2 = 0$, i.e. the integrals are scaleless. Therefore, they vanish in dimensional regularization and with them the purely virtual corrections to $\mathcal{B}_{a/b}$. The same is true for the purely virtual corrections to $\phi$ and $\bar{\mathcal{B}}$.

## 4.6 Remaining Contributions

The remaining contributions, we have to determine, are those from the emission of $n > 0$ partons $\{j_i(l_i)\}_n$. We perform a NNLO calculation and therefore $n = 2$ is the maximal number of emitted partons, we have to consider. This real-real (RR) contribution is the most difficult one, we determine. It is discussed in chapter 7.

For $n = 1$ emitted partons, two types of contributions arise. The squared tree level matrix element $m^{0L}_{a/b,j_1}$ for the emission of a single parton describes the NLO contribution. It will be discussed in chapter 5.

In addition to that, the $n = 1$ case contains a NNLO contribution. The $\alpha_s^2$ term in $|m_{a/b,j_1}|^2$ is

$$|m_{a/b,j_1}|^2_{VR} = m^{1L}_{a/b,j_1} m^{0L\,\dagger}_{a/b,j_1} + m^{0L}_{a/b,j_1} m^{1L\,\dagger}_{a/b,j_1}\,, \tag{4.22}$$

i.e. combination of the tree level matrix element and its one loop correction. This virtual-real (VR) contribution will be discussed in chapter 6.

Before presenting these calculations in detail, we want to point out several further aspects, relevant for all of them, in the next sections. One the one hand this are relations among individual (nT)PDFs, on the other hand, these are the functional dependencies of the results on the relevant mass scales.

## 4.7 Relations among (nT)PDFs

The calculations, advertised in the last section, will determine the bare parton-to-parton PDFs $\phi_{i/j}$ and nTPDFs $\mathcal{B}_{i/j}$, $\bar{\mathcal{B}}_{i/j}$ up to NNLO in perturbation theory for all choices of partons $i, j \in \{g, q, \bar{q}, q', \bar{q}'\}$, where $q'$ denotes a quark of a flavor different form that of $q$.

Not all of those results are independent, but some are connected by symmetries of QCD. The only independent collinear nTPDFs are

$$\begin{aligned} &\mathcal{B}'_{g/g},\, \mathcal{B}'_{g/q}, \\ &\mathcal{B}_{g/g},\, \mathcal{B}_{g/q},\, \mathcal{B}_{q/g}, \\ &\mathcal{B}_{q/q},\, \mathcal{B}_{\bar{q}/q},\, \mathcal{B}_{q'/q},\, \mathcal{B}_{\bar{q}'/q}, \end{aligned} \tag{4.23}$$

where we suppressed the arguments $(z, x_T^2, \mu, v)$. Corresponding relations hold for the PDFs and anti-collinear nTPDFs. Up to NNLO, in addition to that, $\mathcal{B}_{q'/q} = \mathcal{B}_{\bar{q}'/q}$ holds. Most of the relations are between (anti)-quarks of different flavors and are a consequence of flavor symmetry in QCD. Most of them, we already respected by not explicitly introducing $N_f$ different flavors, but only providing a quark $q$ of unspecified (but same) flavor and a quark $q'$ of different flavor. The introduction of a (anti)-quark with a prime will therefore only be relevant in combination with an unprimed (anti)-quark. In addition to the flavor symmetry, QCD also obeys charge conjugation symmetry, which allows us to exchange particles with their antiparticles. We therefore have the following relations

$$\begin{aligned}
&\mathcal{B}_{\bar{q}/g} = \mathcal{B}_{q/g}\,, \quad \mathcal{B}_{g/\bar{q}} = \mathcal{B}_{g/q}\,, \quad \mathcal{B}'_{g/\bar{q}} = \mathcal{B}'_{g/q}\,, \\
&\mathcal{B}_{\bar{q}/\bar{q}} = \mathcal{B}_{q/q}\,, \quad \mathcal{B}_{q/\bar{q}} = \mathcal{B}_{\bar{q}/q}\,, \\
&\mathcal{B}_{q/q'}\,,\ \mathcal{B}_{\bar{q}'/q},\ \mathcal{B}_{\bar{q}/q'} = \mathcal{B}_{q'/q}\,,
\end{aligned} \tag{4.24}$$

where the last line holds up to NNLO. Corresponding relations hold for $\phi$, $\bar{\mathcal{B}}$, $B$ and $I$. While we do not restrict our actual calculation to the subset (4.23), we can use these relations as a check and present only the results of the independent contributions.

## 4.8 Renormalization Scale Dependence

The defining equations of the nTPDFs, such as eqn. (4.1), are in terms of bare fields and depends on the bare coupling $g_s^{(b)}$. Moreover, they contain poles in the dimensional regulator $\epsilon$. To move from the bare to the renormalized coupling we use

$$g_s^{(b)} = Z_{g_s}(\mu)\Big(\frac{\mu^2 e^{\gamma_e}}{4\pi}\Big)^{\frac{\epsilon}{2}} g_s^{(r)} \tag{4.25}$$

with the renormalization constant

$$Z_{g_s}(\mu) = 1 + \sum_{n=1}^{\infty}\left(\frac{\alpha_s(\mu)}{4\pi}\right)^n Z_{g_s}^{(n)}\,, \tag{4.26}$$

the renormalization scale $\mu$ and $\alpha_s = \frac{g_s^2}{4\pi}$. Note that we have to specify a regularization scheme to unambiguously distribute the terms between $Z_{g_s}(\mu)$ and $g_s^{(b)}$ on the RHS of eqn. (4.25). Throughout our calculation we will use the $\overline{\text{MS}}$-scheme. For this reason we introduced the square root of the $\overline{\text{MS}}$-factor $\big(\frac{\mu^2 e^{\gamma_e}}{4\pi}\big)^{\epsilon}$ in that equation. This contains despite $\mu^{2\epsilon}$, which is needed because of dimensional arguments, also some additional constants to avoid occurrences of related terms in the expansion of the final result. In addition to the presence of that factor, the convention in the $\overline{\text{MS}}$-scheme is that $Z_{g_s}^{(n)}$ are pure poles in $\epsilon$ while $g_s^{(r)}$ is free of poles in that regulator.

For our calculation to NNLO it is sufficient to specify $Z_{g_s}^{(1)} = -\frac{1}{2\epsilon}\beta_0$ with $\beta_0$ given in eqn. (C.4). We then can write

$$\frac{\alpha_s^{(b)}(\mu)}{4\pi} = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \left[\frac{\alpha_s^{(r)}(\mu)}{4\pi} + 2Z_{g_s}^{(1)} \left(\frac{\alpha_s^{(r)}(\mu)}{4\pi}\right)^2 + \mathcal{O}\big(\alpha_s^{(r)3}\big)\right] . \tag{4.27}$$

In this way, we will understand $\alpha_s$ as the renormalized coupling constant in the results of the bare functions in the following sections and at NNLO we will obtain the contribution

$$\mathcal{B}_{a/b}^{(2,Z_{g_s})} = -\frac{\beta_0}{\epsilon}\mathcal{B}_{a/b}^{(1)} \tag{4.28}$$

induced by the corresponding NLO result. The $\overline{\text{MS}}$-factor leads to a dependence of the expansion coefficients $\mathcal{B}_{a/b}^{(n)}$ on the renormalization scale $\mu$. These functions still contain poles in $\epsilon$ and require an operator renormalization, as mentioned in section 3.1.6 and explained more explicitly in section 8.2.

Despite the poles in $\epsilon$, the functions also contain poles in the analytic regulator $\alpha$ and a dependence on the associated scale $\upsilon$. Among others, this latter dependence will be discussed in the next subsection.

## 4.9 Dependence on Other Scales

We have just seen how the dependence on the renormalization scale enters our calculation of $\mathcal{B}^{(n)}$. From eqn. (4.6) we can identify the other mass scales, the functions $\mathcal{B}$ can depend on after all integrals have been evaluated. On the one hand, this is the scale $\upsilon$ associated to the analytic regulator. On the other hand, these are scalar products of the vectors $x_\perp$, $n$, $\bar{n}$ as well as $p$ in the collinear case or $\bar{p}$ in the anti-collinear case. Not all of them are independent, since we have

$$p^\mu = \frac{\bar{n}\cdot p}{2} n^\mu \sim n^\mu , \tag{4.29}$$

$$\bar{p}^\mu = \frac{n\cdot \bar{p}}{2} \bar{n}^\mu \sim \bar{n}^\mu . \tag{4.30}$$

Most of their scalar products vanish, since $n$ and $\bar{n}$ and therefore $p$ and $\bar{p}$ are LC vectors and $x_\perp$ is orthogonal to them, i.e.

$$n^2,\, \bar{n}^2 = 0 , \tag{4.31}$$

$$n\cdot x_\perp,\, \bar{n}\cdot x_\perp = 0 . \tag{4.32}$$

Moreover, the scalar product between $n$ and $\bar{n}$ has no mass dependence, but is the simple number $n\cdot\bar{n} = 2$.

Therefore, the only mass scales besides $\upsilon$ and $\mu$ the nTPDFs can depend on, are $x_\perp^2 = -x_T^2$ as well as either $\bar{n}\cdot p$ in the collinear region or $n\cdot\bar{p}$ in the anti-collinear region. The

PDFs, furthermore, do not depend on $x_T^2$.

We will find in the following, that all dependence of the nTPDFs on mass scales can be expressed by the logarithms

$$L_\perp = \log \frac{x_T^2 \mu^2}{4e^{-2\gamma_e}} \ , \tag{4.33}$$

$$L_a = \log \frac{v}{\bar{n}\cdot p} \ , \tag{4.34}$$

$$L_c = \log \frac{v\, n\cdot\bar{p}\, x_T^2}{4e^{-2\gamma_e}} \ , \tag{4.35}$$

and the $\mu$ dependence of the renormalized coupling $\alpha_s^{(r)}(\mu)$. The factor $4e^{-2\gamma_e}$ with the Euler-–Mascheroni constant $\gamma_e$ is chosen for convenience to avoid corresponding logarithms in the expansion.

One can understand the arising of $L_\perp$ from the $d-2$ dimensional integral over $k_\perp$, for which the only relevant scale in eqn. (4.6) is $x_\perp$. This will become clearer, once we discuss the explicit integrals in the following chapters.

The arising of $L_a$ and $L_c$ is a consequence of the $v$-dependence introduced with the analytic regulator $\alpha$ in eqn. (4.2). There it always appears as ratio $v/n\cdot l_i$ with the LC component of an external momentum. For the anti-collinear part this LC component can be expressed by the large component of the momentum of the incoming parton $n\cdot\bar{p}$ multiplied by some dimensionless variable $r_i$. Hence, after the integration, $v$-dependence should always be expressible as $v/\bar{n}\cdot p$, which leads to $L_a$.

In the collinear sector, we can express the LC component of the external momentum in $v/n\cdot l_i = v\,\bar{n}\cdot l_i/l_{i\perp}^2$ by the transverse scale $x_T^2$ and the large component of the incoming parton $\bar{n}\cdot p$ multiplied by some dimensionless variable. Therefore, after integration, the $v$-dependence is expected to be of the form $v\, n\cdot\bar{p}\, x_T^2$, which leads to $L_c$.

Note that no $\mu$ dependence is expected to appear in a ratio with $v$, since $\mu$ never appears with a power of $\alpha$. Similarly, we do not expect the appearance of the two additional logarithms, $\log(\bar{n}\cdot p/\mu)$ in the collinear region and $\log(n\cdot\bar{p}/\mu)$ in the anti-collinear region, since the LC components will be never raised to powers of $\epsilon$, which however is the way the scale $\mu$ appears.

In our (N)NLO results for $\mathcal{B}$ ($\bar{\mathcal{B}}$), we will explicitly observe the functional dependence on $L_\perp$ and $L_c$ ($L_a$) only. Therefore, we can also write them as functions of the arguments $(z, L_\perp, L_{c/a}, \alpha_s)$ instead of $(z, x_T^2, \mu, v)$ and the large momentum component of the incoming partons. This is in fact the way, we will present our results in the following sections. We will use the notations interchangeable without explicit new symbols for the corresponding functions.

The different dependence on $\alpha$ and the associated logarithms in the two regions will be responsible for the cancellation of the poles in $\alpha$ in the product of a collinear and a corresponding anti-collinear nTPDF and lead to the arising of a logarithm of the hard scale $q^2$. We will come back to these points in section 8.1. Now we discuss the perturbative calculations of the remaining NLO and NNLO contributions as mentioned in section 4.6.

# 5 Single Emission, NLO

In this chapter, we discuss the NLO contribution. It is obtained from the single emission at tree level. We will first discuss the corresponding topologies and then the relevant steps to extract the results from them.

## 5.1 Topologies

Let us consider the collinear region in the LC gauge with the LC vector $\bar{n}$. The considerations for the anti-collinear region are implied by exchanging $p \sim n \leftrightarrow \bar{p} \sim \bar{n}$.

With the momentum $p$ of the incoming parton $b$ and the momentum $k$ of the emitted parton $j$, we identify the QCD like matrix element $m_{a/b,j(k)}$ as defined in eqns. (4.5, 4.11, 4.12) at tree level. There is only a single amplitude topology given by figure 6.1(a). The only propagator carries momentum $p-k$. This can lead to a single power of $(p-k)^{-2}$ and of $1/\bar{n}\cdot k$. The squared matrix element $\overline{|M^{(1)}_{a/b}|^2}$ obtained from $m_{a/b,j(k)}$ via eqn. (4.13) can correspondingly contain two powers of each propagator. In addition to the denominators from the normal propagators, it can also contain single powers of the light cone propagators of $k$ and $p$ from the cut propagators of the external gluons as described in sections 4.2 and 4.2.1.

## 5.2 Basic Expression

As follows from eqn. (4.14), at NLO we have to consider the following integrals

$$\frac{\alpha_s}{4\pi}\Big\{\mathcal{B}^{(1)}_{a/b}\,;\,\phi^{(1)}_{a/b}\Big\}(z) = \int \frac{\mathrm{d}^d k}{(2\pi)^{d-1}}\delta^+(k^2)\delta(k_- - z_- p_-)\Big(\frac{v}{k_+}\Big)^\alpha \Big\{e^{ik_T\cdot x_T};\ 1\Big\}\overline{|M^{(1)}_{a/b}|^2}\,. \qquad (5.1)$$

After renaming $p \sim n \leftrightarrow \bar{p} \sim \bar{n}$ for the anti-collinear case, the same expression is obtained there but with the analytic regulator as power of $k_-$. For the gluon nTPDF, the contractions with the second projector gives the same expression, but with $\overline{|M'^{(1)}_{a/b}|^2}$. In this case, the numerator can also contain contractions of $x_\perp$ with any of the momenta.

## 5.3 Notations and Simplifications

The single emission the integrals are not too hard. Using the light-cone components

$$k_+ = n \cdot k \tag{5.2}$$
$$k_- = \bar{n} \cdot k \tag{5.3}$$

we decompose $k^2 = k_+ k_- - \vec{k}_T^2$ as well as

$$\mathrm{d}^d k = \frac{1}{2} \mathrm{d}k_+ \mathrm{d}k_- \mathrm{d}^{d-2} k_T \,. \tag{5.4}$$

Then we can use the two $\delta$-functions to rewrite

$$\int \frac{\mathrm{d}^d k}{(2\pi)^{d-1}} \delta^+(k^2) \delta(k_- - z_- p_-) = \frac{1}{2^{4-2\epsilon} \pi^{3-2\epsilon}} \frac{1}{z_- p_-} \int \mathrm{d}^{d-2} k_T \tag{5.5}$$

and replace

$$k_+ = k_T^2/(z_- p_-) \qquad \text{and} \qquad k_- = z_- p_- \,. \tag{5.6}$$

The integrand is a function of the linearly independent 4-vectors $k^\mu$. $p^\mu$, $\bar{n}^\mu$ and $x_\perp^\mu$. Related to them are

$$q^\mu = p^\mu - k^\mu \quad \text{and} \quad n^\mu = 2p^\mu / p_- . \tag{5.7}$$

We then have

$$p^2 = 0, \qquad \bar{n}^2 = 0, \qquad k^2 = 0, \tag{5.8}$$

for the two light-cone vectors and the massless on-shell momentum. Because $x_\perp$ is perpendicular to $n$ and $\bar{n}$, we also have

$$p \cdot x_\perp = 0, \qquad \bar{n} \cdot x_\perp = 0 \,. \tag{5.9}$$

Moreover we write

$$p \cdot k = -\frac{1}{2}(p-k)^2 = -\frac{1}{2} q^2 = \frac{1}{2} p_- k_+ = \frac{k_T^2}{2 z_-} \,. \tag{5.10}$$

Therefore, each term in $\overline{|M_{a/b}^{(1)}|^2}$ of (5.1) can be written as a product of powers of $k_T^2$, $k_\perp \cdot x_\perp$, $z$, $z_-$ and $p_-$. Of them, $k_\perp \cdot x_\perp$ can only appear for $\mathcal{B}'_{g/b}$ where it is introduced by the corresponding projector $P_x$ in eqn. (3.60). Obviously, it can only occur in the numerator and at most to the second power. Due to eqn. (5.9), it appears either to power 0 or power 2.

## 5.4 $\mathbf{k_T}$-Integral of First Kind

Combining the integral (5.5) with the power of $k_T^2$ and in case of the nTPDF moreover the factor $e^{ik_T \cdot x_T}$ and potential powers of $k_\perp{\cdot}x_\perp$ we can solve it easily. The normal PDF $\phi$ does not know about $x_T$. This leads to scaleless integral

$$\int \mathrm{d}^{2-2\epsilon} k_T (k_T^2)^w = 0 \tag{5.11}$$

which vanishes in dimensional regularization with $\epsilon = \epsilon_{UV} = \epsilon_{IR}$. Hence, we have

$$\phi^{(1)}_{a/b}(z,\mu) = 0\ , \tag{5.12}$$

i.e. we do not find perturbative corrections for the bare normal PDF.

For $\mathcal{B}$ we have the additional factor $e^{ik_T \cdot x_T}$. If no power of $k_\perp{\cdot}x_\perp$ is present, the remaining $k_T$ integral yields

$$\begin{aligned}\int \mathrm{d}^{2-2\epsilon} k_T (k_T^2)^w e^{ik_T\cdot x_T} &= \int_{-\infty}^{\infty} \mathrm{d}k_x e^{ik_x x_T} \int_0^\infty \mathrm{d}k_r\, k_r^{-2\epsilon} \big(k_x^2 + k_r^2\big)^w \int \mathrm{d}\Omega_{1-2\epsilon} \\ &= \pi^{1-\epsilon} \frac{\Gamma(1+w-\epsilon)}{\Gamma(-w)} \left(\frac{x_T^2}{4}\right)^{-1-w+\epsilon}\ ,\end{aligned} \tag{5.13}$$

where we worked in the dimension $d = 4-2\epsilon$ and split $k_T$ in a component $k_x$ parallel to $x_T$ and a component $k_r$ orthogonal to it. For the latter we used spherical coordinates with

$$\int \mathrm{d}\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\ , \tag{5.14}$$

and performed the remaining integrations in Mathematica. With the help of the equations listed in section B.1 the final result can be obtained. The power $w$ we usually find at NLO for eqn. (5.13) is $-1-m\alpha$ with $m=0$ for the anti-collinear and $m=1$ for the collinear case. For both cases the integral leads to a single pole in $\epsilon$ generated by the $\Gamma$-function in the numerator.

## 5.5 $\mathbf{k_T}$-Integral of Second Kind

Due to the presence of $(x_\perp \cdot k)^2$ there will appear a second kind of $k_T$ integral in the calculation of $\mathcal{B}'_{g/b}$. Instead of the integrand in eqn. (5.13) one finds the new integrand

$$(k_T^2)^b \frac{(-x_T{\cdot}k_T)^2}{-x_T^2} e^{ik_T\cdot x_T} = -k_x^2 e^{ik_x x_T} \left(k_x^2 + k_r^2\right)^b = \partial^2_{x_T} e^{ik_x x_T} \left(k_x^2 + k_r^2\right)^b\ . \tag{5.15}$$

Again $k_x$ denotes the component of $k_T$ along $x_T$ while $k_r$ denotes the remaining components orthogonal to it. Exchanging the order of integration and differentiation, eqn. (5.13) implies

$$\int \mathrm{d}^{2-2\epsilon} k_T (k_T^2)^b \frac{(x_T \cdot k_T)^2}{-x_T^2} e^{ik_T \cdot x_T} = \tag{5.16}$$
$$2^{-2}(-3-2b+2\epsilon)(-2-2b+2\epsilon)\pi^{1-\epsilon}\frac{\Gamma(1+b-\epsilon)}{\Gamma(-b)}\left(\frac{x_T^2}{4}\right)^{-2-b+\epsilon}.$$

The power $b$ at NLO is usually $-2-m\alpha$ with $m=0$ for the anti-collinear and $m=1$ for the collinear case. Hence, this integral will lead to a single pole in $\epsilon$ but does not contain a pole in $\alpha$.

## 5.6 Bare Results

Using the relations and solutions of the integrals discussed in this chapter, we can obtain the solution for the NLO (nT)PDFs in a closed form. Since these solutions are needed beyond the finite order in the regulators $\epsilon$ and $\alpha$ and the closed form is very compact, we will provide the exact solution. All results are obtained in dimensional regularization with $\epsilon_{IR}=\epsilon_{UV}=\epsilon$. In addition, the analytic regulator $\alpha$ is used. The dependence on the associated scale $v$ is expressed through $L_a = \log v/n\cdot\bar{p}$ in the anti-collinear region and through $L_c = \log(v\,\bar{n}\cdot p\, x_T^2 e^{2\gamma_e}/4)$ in the collinear region. The nTPDFs, furthermore, depend on the scale $x_T^2$. This dependence is incorporated in $L_\perp = \log(x_T^2\mu^2 e^{2\gamma_e}/4)$.

As discussed in section 4.8, we replace occurrences of the bare coupling constant by the renormalized one. For the NLO contribution single emission this leads to the inclusion of the $\overline{\mathrm{MS}}$-factor $\left(\frac{\mu^2 e^{\gamma_e}}{4\pi}\right)^\epsilon$. In addition to that, each bare NLO result implies a contribution to NNLO via eqn. (4.27) which is given as

$$\mathcal{B}_{i/j}^{(2,Z_{g_s})} = -\frac{\beta_0}{\epsilon}\mathcal{B}_{i/j}^{(1)}\,. \tag{5.17}$$

For the PDF with any partons $i$ and $j$, we obtain the simple result

$$\phi_{i/j}^{(1)}(z,\mu) = 0\,. \tag{5.18}$$

For the nTPDFs, the results are of the form

$$\mathcal{B}_{i/j}^{(1)}(z,x_T^2,\mu,v) = e^{\alpha L_c+\epsilon L_\perp} e^{-(\epsilon+2\alpha)\gamma_e}\frac{\Gamma(-\epsilon-\alpha)}{\Gamma(1+\alpha)}(1-z)^\alpha\, f_{i/j}^{(1)}(z,\epsilon) \tag{5.19}$$

and

$$\bar{\mathcal{B}}_{i/j}^{(1)}(z,x_T^2,\mu,v) = e^{\alpha L_a+\epsilon L_\perp} e^{-\epsilon\gamma_e}\Gamma(-\epsilon)(1-z)^{-\alpha}\, f_{i/j}^{(1)}(z,\epsilon)\,, \tag{5.20}$$

for the collinear and anti-collinear region, respectively, with the functions $f_{i/j}$ given by

$$\begin{aligned}
f^{(1)}_{g/g}(z,\epsilon) &= 4C_a(1-z)^{-1}\Big[\frac{(1-z+z^2)^2}{z}\Big]\,,\\
f^{(1)}_{g/q}(z,\epsilon) &= 2C_f\Big[\frac{1+(1-z)^2}{z}-\epsilon z\Big]\,,\\
f^{(1)}_{q/g}(z,\epsilon) &= 2T_f\Big[1-\frac{2}{1-\epsilon}z(1-z)\Big]\,,\\
f^{(1)}_{q/q}(z,\epsilon) &= 2C_f(1-z)^{-1}\big[2z+(1-\epsilon)(1-z)^2\big]\,,\\
f^{(1)}_{q'/q}(z,\epsilon)\,,\ f^{(1)}_{\bar{q}/q}(z,\epsilon) &= 0.
\end{aligned} \tag{5.21}$$

As pointed out in section 4.7, the other parton combinations are connected by conjugation and flavor symmetry.

For the gluon nTPDF, in addition to that, the second tensor structure exists. The corresponding results are of the form

$$\mathcal{B}'^{(1)}_{g/j}(z,x_T^2,\mu,v) = e^{\alpha L_c+\epsilon L_\perp}e^{-(\epsilon+2\alpha)\gamma_e}\frac{\Gamma(1-\epsilon-\alpha)}{\Gamma(2+\alpha)}(1-z)^\alpha\, g^{(1)}_{g/j}(z) \tag{5.22}$$

and

$$\bar{\mathcal{B}}'^{(1)}_{g/j}(z,x_T^2,\mu,v) = e^{\alpha L_a+\epsilon L_\perp}e^{-\epsilon\gamma_e}\Gamma(1-\epsilon)(1-z)^{-\alpha}\, g^{(1)}_{g/j}(z)\,, \tag{5.23}$$

with the functions

$$\frac{g^{(1)}_{g/g}(z)}{C_a} = \frac{g^{(1)}_{g/q}(z)}{C_f} = 4\,\frac{1-z}{z}\,. \tag{5.24}$$

The expansion of all functions can be easily obtained, as it contains only Γ-functions and powers of constants, $L_\perp$, $L_c$, $L_a$, $z$ and $z_-$. We noted earlier that the expansion of the Γ-functions starts at order $\epsilon^{-1}$ but does not contain poles in $\alpha$, since we have to expand in $\alpha$ first. The analytic regulator is most relevant for the diagonal splittings $\mathcal{B}_{a/a}$. For those we find $z_-^{-1-s\alpha}$ with $s=-1$ for the anti-collinear and $s=1$ for the collinear case. This factor has a pole at $z=1$ which is only regulated by $\alpha$. The expansion is done in terms of distributions via eqn. (B.32) which leads to single poles in $\alpha$. Note that the factors $z_-^y$ are generated via $\delta(k_- - z_-p_-)$ by the integral over the light-cone component $k_-$ of $k$. It is easily seen that the analytic regulator which we are using is well designed to regulate these poles.

While the NLO nTPDFs for the second gluon tensor structure have neither poles in $\alpha$ nor in $\epsilon$, the remaining NLO nTPDFs contain such poles. More precisely, the off-diagonal ones contain single poles in $\epsilon$, but no poles in $\alpha$, while the diagonal ones contain single poles in $\alpha$ and poles up to second order in $\epsilon$, but the sum of the powers of $\epsilon$ and $\alpha$ is at least -2. Due to the terms contained in the expansion of this function, a pole in $\epsilon$ of second order can be generated.

For all functions, the dependence on the mass scales could completely be extracted by the first factors in eqns. (5.19, 5.20, 5.22, 5.23). The logarithms containing $v$ appear together with a power of the corresponding regulator $\alpha$. The exact definition of these logarithms differs between the two regions, as does the dependence on the corresponding regulator.

This different dependence will eventually lead to the cancellation of the $\alpha$- and $v$-dependence between the two regions. We will discuss in detail, how this happens and how the refactorized nTPDFs can be extracted in section 8.1. Before that, we will discuss the calculation of the nTPDFs to the next order in perturbation theory in the next two chapters. We will start with the 1-Loop corrections to the single emission.

# 6 Virtual Correction to Single Emission

Now let us consider the 1-loop virtual correction to the single emission discussed in the last chapter.

We will first discuss the topologies we encounter, then the related basic integral expressions. The appearing integrals can be reduced to a smaller subset, as will be discussed in sections 6.3 and 6.5. The remaining class of integrals is solved in section 6.4.

## 6.1 Topologies

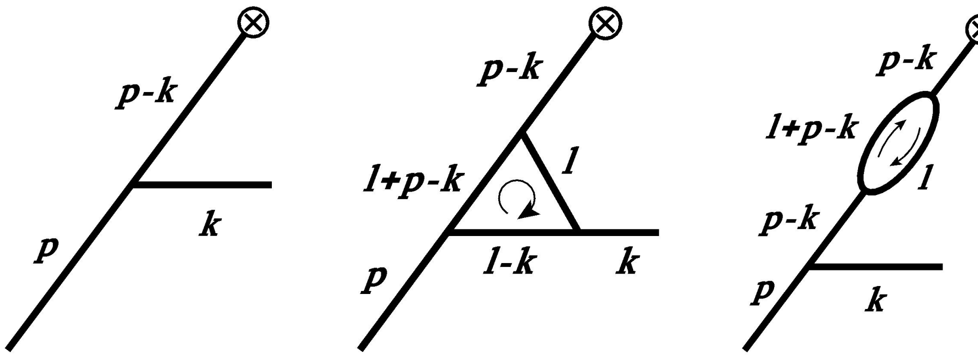
Let us consider the QCD like matrix element $m^{1L}_{a/b,j}$ for the emission of a single parton $j$ of momentum $k$. They are similar to those of the last section, but with an internal loop carrying momentum $l$. The two different amplitude topologies are depicted in Figure 6.1(b,c) - the triangle loop (b) and the propagator loop (c). By shrinking lines to single points, one can obtain further amplitude subtopologies. Shrinking either the left or the right line of the triangle, one receives a bulb on the incoming or the unresolved outgoing parton. By shrinking either one line of the loop or the line entering the loop in the propagator loop diagram, one obtains a snail on the outgoing parton or a loop directly connected to the vertex of the unresolved emission.

With respect to the single emission without 1-loop correction, we find the following additional propagators: $l$ and $l+p-k$ for both amplitude topologies as well as either $l-k$ for the triangle or an additional occurrence of $p-k$ for the propagator loop. Each propagator of momentum $h$ can produce both, a power of $1/h^2$ and of $1/\bar{n}\cdot h$. This implies that the maximal power of both kinds of denominators, constructed from the $l$ dependent propagators $l$, $l+p-k$ and $l-k$, is one. Note that the denominator $1/\bar{n}\cdot h$ is introduced through the light-cone gauge and only appears for propagating gluons of momentum $h$, propagators of (anti)-quarks do not lead to such a denominator.

For the determination of the corresponding NNLO corrections to $\mathcal{B}_{a/b}$ and $\Phi_{a/b}$, we have to combine the matrix element $m^{1L}_{a/b,j}$ discussed here with the corresponding element $m^{0L}_{a/b,j}$ discussed earlier to obtain

$$|m_{a/b,j_1}|^2_{VR} = m^{1L}_{a/b,j_1} m^{0L\,\dagger}_{a/b,j_1} + m^{0L}_{a/b,j_1} m^{1L\,\dagger}_{a/b,j_1} \,. \tag{6.1}$$

Combined with the remaining factors and summed over spin and color, this gives rise to $\overline{|M^{(1)}_{a/b}|^2_{VR}}$ by eqn. (4.13). Then in addition to the denominators containing $l$, we will have those present for the single emission without virtual correction, which has been discussed in section 5.1. Furthermore, for the propagator loop topology, the maximal power of the




(a) (b) (c)

Figure 6.1: Amplitude topologies for real (a) and virtual-real (b,c) case.

two denominators containing $p - k$ is increased by one w.r.t. the NLO contribution.

## 6.2 Basic Expression

For the topologies of $\overline{|M^{(1)}_{\bar{q}/b}|^2}_{VR}$ discussed in the last section, we have to solve the following kind of integrals

$$\left(\frac{\alpha_s}{4\pi}\right)^2 \left\{\mathcal{B}^{(VR)}_{a/b};\, \phi^{(VR)}_{a/b}\right\}(z) = \int \frac{\mathrm{d}^d k}{(2\pi)^{d-1}} \left(\frac{v}{k_+}\right)^{\alpha} \delta^+(k^2)\delta(k_- - z_- p_-)\left\{e^{ik_T\cdot x_T};\ 1\right\} \times \int \frac{\mathrm{d}^d l}{(2\pi)^d} \overline{|M^{(1)}_{a/b}|^2}_{VR}\,, \tag{6.2}$$

where $k$ is the momentum of the emitted particle and $l$ is the loop momentum. Renaming $p \sim n \leftrightarrow \bar{p} \sim \bar{n}$ in the anti-collinear case, the same expression is obtained there but with the analytic regulator as power of $k_-$. For the gluon nTPDF, the contractions with the second projector gives the same expression, but with $\overline{|M'^{(1)}_{a/b}|^2}_{VR}$. In this case, the numerator can also contain contractions of $x_\perp$ with any of the momenta.

Note that the analytic regulators, the exponential containing $k_T$ and the $\delta$-functions do not depend on the loop momentum $l$. Therefore, the $l$ integral is of standard form and can be solved by well established methods. We will do so in the following sections.

Once we noticed there that for each term the potential $k$-dependence will factorize from the result of the $l$-integral, the remaining $k$-integral is just of the same form as we discussed in section 5. In particular, this implies that we can perform them by the $\delta$-functions and eqns. (5.13, 5.16). Hence the only thing left to do for the virtual-real contribution is dealing with the loop integral in eqn. (6.2).

| $(x,x')$ | $(l,p-k)$ | $(l,-k)$ | $(p-k,-k)$ |
|---|---|---|---|
| $b_{xx'}$ | $1/(p-k)_-$ | $-1/k_-$ | $-1/p_-$ |
| $c_{xx'}$ | $-1/(p-k)_-$ | $1/k_-$ | $1/p_-$ |

Table 6.1: Partial fraction parameters for $D_x = \bar{n}\cdot(l+x)$ with $x \in \{0, p-k, -k\}$.

## 6.3 Simplifications

As discussed in section 6.1, we can encounter the $l$ dependent denominators $l^2$, $(l+p-k)^2$ and $(l-k)^2$ as well as $\bar{n}\cdot l$, $\bar{n}\cdot(l+p-k)$ and $\bar{n}\cdot(l-k)$. The later ones only appear if the corresponding propagator is that of a gluon. The maximal powers are 1. These powers can be reduced by the numerators present. The **numerator** can also contain further scalar products containing $l$. Those we express in terms of the existing denominators by expanding the momenta in terms of $l$, $k$ and $p-k$ and then writing

$$v\cdot l = \frac{1}{2}\Big[(l+v)^2 - l^2 - v^2\Big], \tag{6.3}$$

with $v = -k, p-k$. For the second gluon tensor structure also the factor $l\cdot x_\perp$ or its square can appear. These cannot be expanded in terms of the existing denominators. The discussion of integrals with such factors will be given in subsection 6.5. For the discussion following in this section their presence is irrelevant, however, and occurrences of $(l+v)\cdot x_\perp$ will always be expanded in terms of $l\cdot x_\perp$ and a piece independent of $l$.

In the next step we use **partial fraction decomposition** for the three denominators $D_x = (l+x)_-$ with $x \in \{0, p-k, -k\}$. The method is described in section A. The relevant equations are (A.4, A.5, A.6) with the parameters $c_{xx'}, b_{xx'}$ given in Table 6.1. As all $l$-dependent denominators occur only with integer powers, after partial fraction decomposition each term will contain only one of the denominators $(l+x)_-$.

Next, we want to change all of them to $l_-$. If the power of $(l+x)_-$ is positive, we simply expand that term. If we encounter $1/(l+p-k)_-$ or $1/(l-k)_-$, we use the freedom to **shift and rename the integration variable** $l$, in such a way that we obtain $1/l_-$. For terms containing $1/(l+p-k)_-$ we change the integration variable to $l' = -l-p+k$ and rename it to $l$ again. This exchanges $l$ with $-(l+p-k)$ and $l-k$ with $-(l+p)$ and vice versa. Then besides $1/l_-$ the three possible denominators are $(l+x)^2$ with $x \in \{0, p-k, p\}$. In the same way we treat terms free of $1/(l+x)_-$. For terms containing $1/(l-k)_-$ we change and then redefine the integration variable $l$ to $-l+k$. This exchanges $l$ with $-(l-k)$ and $l+p-k$ with $-(l-p)$ and vice versa. Then besides $1/l_-$ the three possible denominators are $(l+x)^2$ with $x \in \{0, p-k, -p\}$. In our calculation, terms containing $1/(l-p)^2$ are free of $1/(l+p-k)^2$, such that we can change $l$ to $-l$ for those terms to obtain $1/(l+p)^2$ again.

Positive powers of $(l+x)^2$ can easily be expanded in terms of the denominators present. Whenever we have the choice, we prefer to remove powers of $(l-k)^2$. Then we only have to consider the two different sets of the four denominators: $l_-$, $l^2$, $(l+p-k)^2$ and either $(l+p)^2$ or $(l-k)^2$. The corresponding integrals are defined in the next section. Note that our earlier discussion implies that each denominator appears with an integer power with the minimum power being $-1$. Negative powers for the three full denominators at once can only appear for the amplitude topology (b) in figure 6.1. For amplitude topology (c) we can never encounter a negative power of $(l+p)^2$ or $(l-k)^2$.

## 6.4 The Scalar 1-Loop Integrals

Due to the steps discussed in the last section to reduce the number of different denominators, we are only left with the following two different types of scalar 1-loop integrals

$$I_1^{\mathrm{VR}}(a_1,a_2,a_3,a_4) = \int \frac{\mathrm{d}^d l}{(2\pi)^d}[-l^2]^{-a_1}[-(l+q)^2]^{-a_2}[-(l+p)^2]^{-a_3}[\bar{n}\cdot l]^{-a_4} \ , \tag{6.4}$$

$$I_2^{\mathrm{VR}}(a_1,a_2,a_3',a_4) = \int \frac{\mathrm{d}^d l}{(2\pi)^d}[-l^2]^{-a_1}[-(l+q)^2]^{-a_2}[-(l-k)^2]^{-a_3'}[\bar{n}\cdot l]^{-a_4} \ , \tag{6.5}$$

with $q = p + k$ and integer values of the $a_i$. In all propagators an imaginary part $-i\delta$ is implicit. These integrals can be solved by standard methods. We use Feynman parameters to rewrite the integrand and then easily perform the integral over the loop-momenta. For the first integral for example, one obtains

$$\begin{aligned} I_1^{\mathrm{VR}}(a_1,a_2,a_3,a_4) = &\frac{i}{2^{4-2\epsilon}\pi^{2-\epsilon}}\frac{\Gamma(a_1+a_2+a_3+a_4-2+\epsilon)}{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(a_3)\,\Gamma(a_4)}\int_0^1 \mathrm{d}x_2\int_0^1 \mathrm{d}x_3\int_0^\infty \mathrm{d}x_1\int_0^\infty d\lambda \\ &\times \delta(1-x_2-x_3)x_1^{a_1-1}\,x_2^{a_2-1}\,x_3^{a_3-1}\,\lambda^{a_4-1}(x_1+x_2+x_3)^{a_1+a_2+a_3+a_4-4+2\epsilon} \\ &\times \left[-q^2x_1x_2 - \bar{n}\cdot(x_2q+x_3p)\,\lambda\right]^{2-\epsilon-a_1-a_2-a_3-a_4} \ . \end{aligned} \tag{6.6}$$

The remaining integrals can be carried out in a closed form in terms of Hypergeometric functions. We use Mathematica to do this. The solutions are given by

$$\begin{aligned} I_1^{\mathrm{VR}}(a_1,a_2,a_3,a_4) = &\frac{i}{2^{4-2\epsilon}\pi^{2-\epsilon}}(-q^2)^{2-\epsilon-a_1-a_2-a_3}(-p_-)^{-a_4} \\ &\times \frac{\Gamma(-2+\epsilon+a_1+a_2+a_3)\Gamma(2-\epsilon-a_1-a_3)\Gamma(2-\epsilon-a_2-a_3)\Gamma(2-\epsilon-a_1-a_4)}{\Gamma(a_1)\Gamma(a_2)\Gamma(2-\epsilon-a_1)\Gamma(4-2\epsilon-a_1-a_2-a_3-a_4)} \\ &\times {}_2F_1(a_4, 2-\epsilon-a_1-a_3; 2-\epsilon-a_1; z_-) \ , \end{aligned} \tag{6.7}$$

$$\begin{aligned} I_2^{\mathrm{VR}}(a_1,a_2,a_3,a_4) = &\frac{i}{2^{4-2\epsilon}\pi^{2-\epsilon}}(-q^2)^{2-\epsilon-a_1-a_2-a_3}(z_-p_-)^{-a_4} \\ &\times \frac{\Gamma(-2+\epsilon+a_1+a_2+a_3)\Gamma(2-\epsilon-a_1-a_3)\Gamma(2-\epsilon-a_2-a_3)\Gamma(2-\epsilon-a_1-a_4)}{\Gamma(a_1)\Gamma(a_2)\Gamma(2-\epsilon-a_1)\Gamma(4-2\epsilon-a_1-a_2-a_3-a_4)} \\ &\times {}_2F_1(a_4, 2-\epsilon-a_1-a_3; 2-\epsilon-a_1; 1/z_-) \ , \end{aligned} \tag{6.8}$$

where ${}_2F_1$ are Hypergeometric functions, discussed in section B.3 (see e.g. eqn. (B.25)) and we use the shorthand notation $z_- = 1 - z$.

In the next subsection we will relate the analogous integrals with powers of $l \cdot x_\perp$ to these expressions and then discuss the expansion of the last two equations in terms of the regulators in section 6.6.

## 6.5 Reduction of Tensor Integrals

In case of the second gluon tensor structure, the integrands of eqns. (6.4, 6.5) can be multiplied by $x_\perp^\mu l_\mu$ or $x_\perp^\mu x_\perp^\nu l_\mu l_\nu$. To take account of these integrals we generalize the integrals of eqns. (6.4, 6.5) once to 1-tensor integrals by introducing the factor $l_\mu$ in the integrand, and once to 2-tensor integrals by introducing the factor $l_\mu l_\nu$ in the integrand. For all of them we will use tensor decomposition to express the new integrals we are actually interested in in terms of a sum of scalar integrals multiplying a choice of independent $n$-tensors relevant for the integral.

### 6.5.1 1-Tensor Integrals

Let us start with the 1-tensor integrals

$$I_{i,\mu}^{\mathrm{VR}}(a_1,a_2,a_3,a_4) = \int \frac{\mathrm{d}^d l}{(2\pi)^d} [-l^2]^{-a_1} [-(l+q)^2]^{-a_2} [-(l+v_i)^2]^{-a_3} [\bar{n}\cdot l]^{-a_4} l_\mu \ , \tag{6.9}$$

for $i = 1, 2$ with $v_1 = p$ and $v_2 = -k$. They can be expressed as a sum of scalar integrals multiplying a choice of independent vectors relevant for the integral. As we are only interested in the contraction of the integral with $x_\perp^\mu$ and we want to make use of eqns. (5.13, 5.16) for the $k_T$-integral, we choose the three independent vectors $k_\perp$, $p$ and $\bar{n}$. This is a valid choice for both integrals due to the denominators appearing in them as well as eqns. (5.7, 4.29, 2.21). Due to Lorentz invariance, we then can write

$$I_{i,\mu}^{\mathrm{VR}}(a_1,a_2,a_3,a_4) = \Big[ k_{\perp\mu}\, I_{i,k_\perp}^{\mathrm{VR}} + p_\mu\, I_{i,p}^{\mathrm{VR}} + \bar{n}_\mu\, I_{i,\bar{n}}^{\mathrm{VR}} \Big](a_1,a_2,a_3,a_4) \ , \tag{6.10}$$

where $I_{i,k_\perp}^{\mathrm{VR}}(a_1,a_2,a_3,a_4)$ and the two other functions on the RHS can be expressed in terms of the scalar integrals $I_i^{\mathrm{VR}}(a_1,a_2,a_3,a_4)$ with adjusted indices. Because scalar products of $x_\perp$ and $k_\perp$ with $p$ and $\bar{n}$ vanish, we have

$$x_\perp^\mu\, I_{i,\mu}(a_1,a_2,a_3,a_4) = k_\perp{\cdot}x_\perp\, I_{i,k_\perp}^{\mathrm{VR}}(a_1,a_2,a_3,a_4)\,, \tag{6.11}$$

$$k_\perp^\mu\, I_{i,\mu}(a_1,a_2,a_3,a_4) = k_\perp^2\, I_{i,k_\perp}^{\mathrm{VR}}(a_1,a_2,a_3,a_4)\,. \tag{6.12}$$

Hence, for our purpose it is sufficient to determine $I_{i,k_\perp}^{\mathrm{VR}}(a_1,a_2,a_3,a_4)$ from the last equation. Plugging in the definition of $I_{i,\mu}(a_1,a_2,a_3,a_4)$, the RHS of this equation will contain the factor $k_\perp^\mu l_\mu$ which has to be rewritten in terms of the denominators appearing in the integral.

In a first step, we can write

$$k_\perp^\mu l_\mu = k \cdot l - \frac{k_-}{p_-} p \cdot l - \frac{k_+}{2} \bar{n} \cdot l \,. \tag{6.13}$$

The last term is already of the requested form. For the other terms, we can use

$$v \cdot l = \frac{1}{2} \Big\{ - \big[ - (l+v)^2 \big] + \big[ - l^2 \big] - v^2 \Big\} \,. \tag{6.14}$$

The relevant choices of $v$ and the resulting expressions depend on the index $i$.

Let us first consider $i = 1$. Then we write $k \cdot l = p \cdot l - q \cdot l$ and use eqn. (6.14) once with $v = p$ and once with $v = q$. With $p^2 = 0$ this leads to

$$k_\perp^\mu l_\mu = \frac{1}{2} \left[ - \frac{k_-}{p_-} \big[ - l^2 \big] + \big[ - (l+q)^2 \big] + \left( \frac{k_-}{p_-} - 1 \right) \big[ - (l+p)^2 \big] - k_+ \big[ \bar{n} \cdot l \big] + q^2 \right] , \tag{6.15}$$

which in turn implies

$$x_\perp^\mu \, I_{1,\mu}^{\mathrm{VR}}(a_1, a_2, a_3, a_4) = \frac{k_\perp \cdot x_\perp}{2 k_\perp^2} \left[ - \frac{k_-}{p_-} \, 1^- + 2^- + \left( \frac{k_-}{p_-} - 1 \right) 3^- - k_+ \, 4^- + q^2 \right] I_1^{\mathrm{VR}}(a_1, a_2, a_3, a_4) \,. \tag{6.16}$$

In the last equation, we have used the lowering operators $n^-$ which act on the $n$th argument of the following function and lower its value by 1.

Analogously the case $i = 2$ can be discussed. Starting again from eqn. (6.13), we now rewrite $p = q + k$ and use eqn. (6.14) once for $v = q$ and once for $v = -k$. With $k^2 = 0$, we then obtain

$$k_\perp^\mu l_\mu = \frac{1}{2} \left[ - \big[ - l^2 \big] + \frac{k_-}{p_-} \big[ - (l+q)^2 \big] + \left( 1 - \frac{k_-}{p_-} \right) \big[ - (l+p)^2 \big] - k_+ \big[ \bar{n} \cdot l \big] + \frac{k_-}{p_-} q^2 \right] , \tag{6.17}$$

which implies

$$x_\perp^\mu \, I_{2,\mu}^{\mathrm{VR}}(a_1, a_2, a_3, a_4) = \frac{k_\perp \cdot x_\perp}{2 k_\perp^2} \left[ - 1^- + \frac{k_-}{p_-} \, 2^- + \left( 1 - \frac{k_-}{p_-} \right) 3^- - k_+ \, 4^- + \frac{k_-}{p_-} \, q^2 \right] I_2^{\mathrm{VR}}(a_1, a_2, a_3, a_4) \,. \tag{6.18}$$

By eqns. (6.16, 6.18) we related the 1-tensor 1-loop integrals back to the scalar 1-loop integrals which have been solved earlier. The remaining $k_T$-integration involving $(k_\perp \cdot x_\perp)^2$ can be performed by means of eqn. (5.16). Hence, we managed to solve the 1-tensor 1-loop integrals. We now turn to the 2-tensor 1-loop integrals

### 6.5.2 2-Tensor Integrals

Analogously to the 1-tensor integrals, we write the 2-tensor integrals as

$$I^{\mathrm{VR}}_{i,\mu\nu}(a_1,a_2,a_3,a_4) = \int \frac{\mathrm{d}^d l}{(2\pi)^d}[-l^2]^{-a_1}[-(l+q)^2]^{-a_2}[-(l+v_i)^2]^{-a_3}[\bar{n}\cdot l]^{-a_4} l_\mu l_\nu \ , \tag{6.19}$$

for $i=1,2$ with $v_1 = p$ and $v_2 = -k$. They can be expressed as a sum of scalar integrals multiplying a choice of independent 2-tensors relevant for the integral. As we are only interested in the contraction of the integrals with $x_\perp^\mu x_\perp^\nu$ and we want to make use of eqns. (5.13, 5.16) for the $k_T$-integral, we decompose $I^{\mathrm{VR}}_{i,\mu\nu}(a_1,a_2,a_3,a_4)$ in terms of $g^\perp_{\mu\nu}$ and the 2-tensors one can construct from $k_\perp$, $p$ and $\bar{n}$. This is a valid choice for both integrals due to the denominators appearing in them as well as eqns. (5.7, 4.29, 2.21). Due to Lorentz invariance, we then can write

$$I^{\mathrm{VR}}_{i,\mu\nu}(a_1,a_2,a_3,a_4) = g^\perp_{\mu\nu}\, I^{\mathrm{VR}}_{i,g_\perp}(a_1,a_2,a_3,a_4) \ + \sum_{v,v'\in\{k_\perp,p,\bar{n}\}} v_\mu v'_\nu \, I^{\mathrm{VR}}_{i,vv'}(a_1,a_2,a_3,a_4)\ . \tag{6.20}$$

From now on, we will suppress the indices $(a_1,a_2,a_3,a_4)$. The integrals $I^{\mathrm{VR}}_{i,g_\perp}$ and $I^{\mathrm{VR}}_{i,vv'}$, can again be related to the scalar integrals $I^{\mathrm{VR}}_i$ with adjusted indices. Because $p\cdot x_\perp$, $\bar{n}\cdot x_\perp = 0$, we obtain for the contraction of interest to us

$$x_\perp^\mu x_\perp^\nu \, I^{\mathrm{VR}}_{i,\mu\nu} = x_\perp^2 I^{\mathrm{VR}}_{i,g_\perp} + (k_\perp\cdot x_\perp)^2 I^{\mathrm{VR}}_{i,k_\perp k_\perp}\ . \tag{6.21}$$

Hence the only two integrals relevant for us are $I^{\mathrm{VR}}_{i,g_\perp}$ and $I^{\mathrm{VR}}_{i,k_\perp k_\perp}$. To obtain them, we consider

$$g_\perp^{\mu\nu}\, I^{\mathrm{VR}}_{i,\mu\nu} = [d-2] I^{\mathrm{VR}}_{i,g_\perp} + k_\perp^2 I^{\mathrm{VR}}_{i,k_\perp k_\perp}\ , \tag{6.22}$$

$$k_\perp^\mu k_\perp^\nu \, I^{\mathrm{VR}}_{i,\mu\nu} = k_\perp^2 I^{\mathrm{VR}}_{i,g_\perp} + (k_\perp^2)^2 I^{\mathrm{VR}}_{i,k_\perp k_\perp}\ , \tag{6.23}$$

where we used $k_\perp\cdot p$, $k_\perp\cdot\bar{n} = 0$. This linear system can be solved easily for the unknowns $I^{\mathrm{VR}}_{i,g_\perp}$ and $I^{\mathrm{VR}}_{i,k_\perp k_\perp}$:

$$I^{\mathrm{VR}}_{i,g_\perp} = \frac{k_\perp^2\big(g_\perp^{\mu\nu}\, I^{\mathrm{VR}}_{i,\mu\nu}\big) - \big(k_\perp^\mu k_\perp^\nu\, I^{\mathrm{VR}}_{i,\mu\nu}\big)}{k_\perp^2[d-3]}\ , \tag{6.24}$$

$$I^{\mathrm{VR}}_{i,k_\perp k_\perp} = \frac{[d-2]\big(k_\perp^\mu k_\perp^\nu\, I^{\mathrm{VR}}_{i,\mu\nu}\big) - k_\perp^2\big(g_\perp^{\mu\nu}\, I^{\mathrm{VR}}_{i,\mu\nu}\big)}{(k_\perp^2)^2[d-3]}\ . \tag{6.25}$$

Thus, as soon as we relate the contracted integrals on the RHS back to the scalar integrals (6.4, 6.5), we have solved our problem. To this end we have to rewrite $g_\perp^{\mu\nu} l_\mu l_\nu$ and $k_\perp^\mu k_\perp^\nu l_\mu l_\nu$ in terms of the denominators appearing in each integral.

For the latter we can just square eqn. (6.15) and (6.17) respectively which in analogy to the 1-tensor case imply

$$k_\perp^\mu k_\perp^\nu\, I^{\rm VR}_{1,\mu\nu}(a_1,a_2,a_3,a_4) = \tag{6.26}$$
$$\frac{1}{2^2}\left[-\frac{k_-}{p_-}\,1^- + 2^- + \left(\frac{k_-}{p_-}-1\right)3^- - k_+\,4^- + q^2\right]^2 I_1^{\rm VR}(a_1,a_2,a_3,a_4)\,,$$
$$k_\perp^\mu k_\perp^\nu\, I^{\rm VR}_{2,\mu\nu}(a_1,a_2,a_3,a_4) = \tag{6.27}$$
$$\frac{1}{2^2}\left[-1^- + \frac{k_-}{p_-}\,2^- + \left(1-\frac{k_-}{p_-}\right)3^- - k_+\,4^- + \frac{k_-}{p_-}\,q^2\right]^2 I_2^{\rm VR}(a_1,a_2,a_3,a_4)\,.$$

For $g_\perp^{\mu\nu} l_\mu l_\nu = l_\perp^2$ we write

$$g_\perp^{\mu\nu} l_\mu l_\nu = -\big[-l^2\big] - \frac{2}{p_-}(p\cdot l)\big[\bar n\cdot l\big]\,, \tag{6.28}$$

and then consider both cases $i=1,2$ separately. For $i=1$ we substitute $p\cdot l$ via eqn. (6.14) with $v=p$ to receive

$$g_\perp^{\mu\nu} l_\mu l_\nu = -\big[-l^2\big] + \frac{1}{p_-}\big[\bar n\cdot l\big]\Big(-\big[-l^2\big] + \big[-(l+p)^2\big]\Big)\,, \tag{6.29}$$

implying

$$g_\perp^{\mu\nu}\, I^{\rm VR}_{1,\mu\nu}(a_1,a_2,a_3,a_4) = \left[-1^- + \frac{1}{p_-}\big(-1^- + 3^-\big)4^-\right] I_1^{\rm VR}(a_1,a_2,a_3,a_4)\,. \tag{6.30}$$

For $i=2$ we write $p\cdot l = q\cdot l + k\cdot l$ and use eqn. (6.14) once with $v=q$ and once with $v=k$ to obtain

$$g_\perp^{\mu\nu} l_\mu l_\nu = -[-l^2] + \frac{1}{p_-}[\bar n\cdot l]\Big(\big[-(l+q)^2\big] - \big[-(l-k)^2\big] + q^2\Big)\,, \tag{6.31}$$

implying

$$g_\perp^{\mu\nu}\, I^{\rm VR}_{2,\mu\nu}(a_1,a_2,a_3,a_4) = \left[-1^- + \frac{1}{p_-}\big(2^- - 3^- + q^2\big)4^-\right] I_2^{\rm VR}(a_1,a_2,a_3,a_4)\,. \tag{6.32}$$

With the results of eqns. (6.26, 6.27, 6.30, 6.32), $I^{\rm VR}_{i,g_\perp}$ and $I^{\rm VR}_{i,k_\perp k_\perp}$ can be obtained straightforwardly from eqns. (6.24) and (6.25) respectively. These results in turn can be plugged into eqn. (6.21) to receive the solutions for the two remaining VR-integrals. With $4^-_{p_-} = \frac{4^-}{p_-}$,

$4^-_{k_-} = \frac{4^-}{[1-z]p_-}$ and $z_- = \frac{k_-}{p_-}$ they read

$$
\begin{aligned}
&x_\perp^\mu x_\perp^\nu \, I^{\mathrm{VR}}_{1,\mu\nu}(a_1,a_2,a_3,a_4) = \\
&\quad\Bigg[\frac{(k_\perp{\cdot}x_\perp)^2[d-2] - x_\perp^2 k_\perp^2}{4[d-3](k_\perp^2)^2}\Big\{z_-(3^- - 1^-) + (2^- - 3^-) + k_\perp^2 z_-^{-1}\big(1 + 4^-_{p_-}\big)\Big\}^2 \\
&\quad + \frac{(k_\perp{\cdot}x_\perp)^2 - x_\perp^2 k_\perp^2}{[d-3]k_\perp^2}\Big\{1^- + (1^- - 3^-)4^-_{p_-}\Big\}\Bigg] I_1^{\mathrm{VR}}(a_1,a_2,a_3,a_4)\ ,
\end{aligned}
\tag{6.33}
$$

$$
\begin{aligned}
&x_\perp^\mu x_\perp^\nu \, I^{\mathrm{VR}}_{2,\mu\nu}(a_1,a_2,a_3,a_4) = \\
&\quad\Bigg[\frac{(k_\perp{\cdot}x_\perp)^2[d-2] - x_\perp^2 k_\perp^2}{4[d-3](k_\perp^2)^2}\Big\{(3^- - 1^-) + z_-(2^- - 3^-) + k_\perp^2\big(1 + 4^-_{k_-}\big)\Big\}^2 \\
&\quad + \frac{(k_\perp{\cdot}x_\perp)^2 - x_\perp^2 k_\perp^2}{[d-3]k_\perp^2}\Big\{1^- + z_-(3^- - 2^-)4^-_{k_-} - k_\perp^2 4^-_{k_-}\Big\}\Bigg] I_2^{\mathrm{VR}}(a_1,a_2,a_3,a_4)\ .
\end{aligned}
\tag{6.34}
$$

In this and the last section, we managed to reduce all $l$-integrals containing contractions with $x_\perp$ to sums of the scalar integrals in eqns. (6.4, 6.5). In the next section, we will discuss the expansion of those integrals.

## 6.6 Expansion of Result

We now discuss the expansion of the RHS of eqns. (6.4, 6.5) in terms of the regulator $\epsilon$. Note that the parameters $a_i$ are integers and these equations do not dependent on $\alpha$. The discussion of section 6.1 implies that $a_i \leq 1$. Given the algorithm of section 6.3 to reduce the denominators, $a_3 = 1$ can only occur for the first VR-topology. The minimal values of $a_i$ appearing are $-3$.

The only two $\Gamma$-functions which are not regulated by $\epsilon$ are $\Gamma(a_1)\Gamma(a_2)$ which appear in the denominator. Therefore, the integrals vanish if $a_1$ or $a_2 \leq 0$. For other cases those two $\Gamma$-functions lead to the factor $[(a_1-1)!(a_2-1)!]^{-1}$ and the remaining ones can easily be expanded in $\epsilon$. Since $a_4$ is an integer and $-q^2 = +k_T^2/z_-$, also the expansion of the factors in the first line of eqns. (6.4, 6.5) is done easily.

Hence, the only remaining pieces are the Hypergeometric functions ${}_2F_1$. Their properties are discussed in section B.3. Note that for the Hypergeometric functions appearing here, the second and third arguments are shifted by $-\epsilon$ from the integers, while the first argument is an integer. If $a_4 \leq 0$, the corresponding ${}_2F_1(a_4,\ldots,x)$ is a polynomial of degree $-a_4$ in its variable $x$, as given in eqn. (B.16). For $a_4 = 1$ there exists another simple case when $a_3 = 0$. Then the second and third index of the Hypergeometric function are equal and positive. Hence, by eqn. (B.15) those indices can be removed and we obtain the simple function ${}_1F_0(1,x) = \frac{1}{1-x}$, where we used eqn. (B.23) in the last step.

The only non-trivial Hypergeometric functions have the indices $a_1, a_2, a_4 = 1$ and $a_3 \neq 0$.

As discussed before eqn. (B.17), the radius of convergence for non-trivial Hypergeometric functions is 1. Because the variable $z_-$ is in $[0,1]$, the non-trivial Hypergeometric functions with the argument $1/z_-$ have to be mapped to two Hypergeometric functions with argument $z_-$ by using the identity (B.18) with $x = z_-^{-1}$ and the assumption of a small imaginary part of $z_-$. To also guarantee the convergence at $z_- = 1$, $\Re(\gamma) > 0$ is required with $\gamma$ defined in eqn. (B.17). If $\Re(\gamma) < 0$, we use eqn. (B.19) to be left with Hypergeometric functions with nice convergence properties which can be easily expanded in Mathematica with the help of the package HypExp.

## 6.7 Final Steps

In the last sections, we have seen, how to solve all appearing integrals over the loop momentum $l$. We now discuss the remaining steps to solve eqn. (6.2). From eqns. (6.4, 6.5) and eqn. (5.10) it is evident that each term in the result of a loop integral depends on $k_\perp$ only via factors $(k_T^2)^{n-\epsilon}$ and $(k_\perp{\cdot}x_\perp)^m$ with integers $n \geq -1$ and $m = 0,\ 1,\ 2$. Combining those powers with the factors not included in the $l$-integral we can perform the remaining $k$-integral in eqn. (6.2) in the same way as at NLO, i.e. we first use eqn. (5.5) and then for $\phi$ eqn. (5.11) or for $\mathcal{B}$ and $\mathcal{B}'$ one of eqns. (5.13, 5.16) with $w = -1 - \epsilon - m\alpha$ and $b = -2 - \epsilon - m\alpha$, respectively. Here $m = 1$ for the collinear case and $m = 0$ for the anti-collinear case. Single powers of $k_\perp{\cdot}x_\perp$ cannot appear for the same reason as at NLO. Note that again corrections to the normal PDF $\phi$ vanish. Appearances of $x_T^2$ in the result will be replaced in terms of $L_\perp$ via eqn. (4.33). Occurrences of the analytic regulator scale $v$ will be expressed via eqn. (4.35) by $L_c$ in the collinear region and via eqn. (4.34) by $L_a$ in the anti-collinear region.

The results of the $k_T$-integral can be expanded in the regulators easily. Both expansions start at $\epsilon^{-1}$, where the pole is introduced through the $\Gamma$-function in the numerator. Poles in $\alpha$ do appear neither through the $k_T$-integral nor through the $l$-integral. However, they can arise through the integral over $k_-$, which can lead to $z_-^{-1+r}$. While for most terms $r = \epsilon + (2m-1)\alpha$ which does not lead to a pole in $\alpha$, we can find $r = (2m-1)\alpha$ for $\mathcal{B}_{a/a}$, i.e. if the resolved outgoing parton is the same as the incoming one. In case of the gluons only the first tensor structure will lead to poles in $\alpha$. As above, $m = 1$ for the collinear case and $m = 0$ for the anti-collinear case.

As discussed in section 4.8, we replace occurrences of the bare coupling constant by the renormalized one. For the VR-contribution to the $\alpha_s^2$ term this leads to the inclusion of the squared $\overline{\text{MS}}$-factor $\left(\frac{\mu^2 e^{\gamma_e}}{4\pi}\right)^{2\epsilon}$. Together with the rest of the expression this factor will be expanded in $\epsilon$ and in this way cancel occurrences of $\gamma_e$.

## 6.8 Bare Results

From the calculation, outlined in the last sections, we extract the results for the virtual-real contribution to the (nT)PDFs. The results are needed to $\alpha^0$, $\epsilon^0$ and in general contain poles in these regulators.

For the PDFs with any partons $i$ and $j$, we again find the simple result

$$\phi_{i/j}^{(VR)}(z,\mu) = 0\,. \tag{6.35}$$

To present the results for the nTPDFs in a compact form, we extract the mass dependent logarithms $L_\perp$, $L_a$ and $L_c$ defined in eqns. (4.33, 4.34, 4.35). We then write the results as

$$\mathcal{B}_{i/j}^{(VR)}(z,x_T^2,\mu,v) = \exp[\alpha L_c + 2\epsilon L_\perp]\, f_{i/j}^{VR}(z,\epsilon,\alpha,1) + \mathcal{O}(\alpha,\epsilon) \tag{6.36}$$

and

$$\bar{\mathcal{B}}_{i/j}^{(VR)}(z,x_T^2,\mu,v) = \exp[\alpha L_a + 2\epsilon L_\perp]\, f_{i/j}^{VR}(z,\epsilon,\alpha,-1) + \mathcal{O}(\alpha,\epsilon)\,. \tag{6.37}$$

The functions $f_{i/j}^{VR}$ can be expressed in terms of harmonic polylogarithms $H_{\vec{a}_n} \equiv H_{\vec{a}_n}(z)$ up to weight 3, $\zeta$-values up to weight 4 and functions $\tilde{p}_{ij}$ related to the lowest order DGLAP splitting kernels $P_{ij}^{(0)}(z)$ by removing an overall factor and the $\delta$-function. These functions are discussed in sections B.2 and C.3, respectively.

$$\begin{aligned} f_{g/g}^{VR}(z,\epsilon,\alpha,s) &= C_a^2\Big\{\delta(1-z)\Big[\frac{4s}{\alpha\epsilon^3} - \frac{4s}{\alpha\epsilon}\zeta_2 + \frac{8s}{3\alpha}\zeta_3 - \frac{1+s}{\epsilon^4} + \frac{3+3s}{\epsilon^2}\zeta_2 + \frac{28(1+s)}{3\epsilon}\zeta_3 \\ &+ \frac{57(1+s)}{4}\zeta_4\Big] + \tilde{p}_{gg}(z)\Big[\frac{4}{\epsilon^3} + \frac{8}{\epsilon^2}H_0 - \frac{4}{\epsilon}\Big(2H_{1,0} + 2H_{0,1} + \zeta_2\Big) + \Big(16H_{1,1,0} + 8H_{1,0,1} + 8H_{0,1,1} \\ &+ 8H_1\zeta_2 + 8H_0\zeta_2 - \frac{16}{3}\zeta_3\Big)\Big] + \Big[\frac{2z}{3\epsilon} + \Big(-\frac{2z}{3}H_1 + \frac{2(6-6z+11z^2)}{9z}\Big)\Big]\Big\} \\ &+ C_aT_fN_f\Big\{\Big[-\frac{4z}{3\epsilon} + \Big(\frac{4z}{3}H_1 - \frac{8(3-3z+7z^2)}{9z}\Big)\Big]\Big\}, \end{aligned} \tag{6.38}$$

$$\begin{aligned} f_{g/q}^{VR}(z,\epsilon,\alpha,s) &= C_fC_a\Big\{\tilde{p}_{gq}(z)\Big[-\frac{2}{\epsilon^3} + \frac{1}{\epsilon^2}\Big(4H_1 + 4H_0 - \frac{22}{3}\Big) + \frac{1}{\epsilon}\Big(-4H_{1,1} - 4H_{1,0} - 4H_{0,1} \\ &+ \frac{22}{3}H_1 - 6\zeta_2 - \frac{152}{9}\Big) + \Big(4H_{1,1,1} + 8H_{1,1,0} + 4H_{1,0,1} + 4H_{0,1,1} - \frac{22}{3}H_{1,1} + 8H_1\zeta_2 + 4H_0\zeta_2 \\ &+ \frac{152}{9}H_1 - \frac{16}{3}\zeta_3 - \frac{22}{3}\zeta_2 - \frac{916}{27}\Big)\Big] + \Big[\frac{2z}{\epsilon^2} + \frac{2z}{\epsilon}\Big(-2H_1 - 2H_0 + \frac{11}{3}\Big) + 2z\Big(2H_{1,1} + 2H_{1,0} + 2H_{0,1} \\ &- \frac{11}{3}H_1 + 3\zeta_2 + \frac{76}{9}\Big)\Big]\Big\} + C_f^2\Big\{\tilde{p}_{gq}(z)\Big[\frac{4}{\epsilon^3} + \frac{1}{\epsilon^2}\Big(-4H_1 + 6\Big) + \frac{1}{\epsilon}\Big(4H_{1,1} - 6H_1 + 4\zeta_2 + 16\Big) \\ &+ \Big(-4H_{1,1,1} + 6H_{1,1} - 4H_1\zeta_2 - 16H_1 + \frac{8}{3}\zeta_3 + 6\zeta_2 + 32\Big)\Big] + \Big[-\frac{4z}{\epsilon^2} + \frac{2z}{\epsilon}\Big(2H_1 - 3\Big) \\ &+ 2z\Big(-2H_{1,1} + 3H_1 - 2\zeta_2 - 8\Big)\Big]\Big\} + C_fT_fN_f\Big\{\tilde{p}_{gq}(z)\Big[\frac{8}{3\epsilon^2} + \frac{1}{\epsilon}\Big(-\frac{8}{3}H_1 + \frac{40}{9}\Big) \\ &+ \Big(\frac{8}{3}H_{1,1} - \frac{40}{9}H_1 + \frac{8}{3}\zeta_2 + \frac{224}{27}\Big)\Big] + \Big[-\frac{8z}{3\epsilon} + \Big(\frac{8z}{3}H_1 - \frac{40z}{9}\Big)\Big]\Big\}, \end{aligned} \tag{6.39}$$

$$f_{q/g}^{VR}(z,\epsilon,\alpha,s) = C_aT_f\Big\{\tilde{p}_{qg}(z)\Big[\frac{2}{\epsilon^3} + \frac{2}{\epsilon^2}\Big(-2H_1 + 1\Big) + \frac{2}{\epsilon}\Big(2H_{1,1} - 2H_{1,0} - 2H_1 + \zeta_2 + 1\Big) \tag{6.40}$$

$$
\begin{aligned}
&+\Big(-4H_{1,1,1}+8H_{1,1,0}+4H_{1,0,1}+4H_{1,1}-4H_{1,0}-4H_1+\frac{4}{3}\zeta_3+2\zeta_2+2\Big)\Big]\\
&+\Big[-\frac{2}{\epsilon^2}+\frac{2}{\epsilon}\Big(2H_1-1-z\Big)+\Big(-4H_{1,1}+4H_{1,0}+2(2+z)H_1-2\zeta_2-6z\Big)\Big]\Big\}\\
&+C_fT_f\Big\{\tilde{p}_{qg}(z)\Big[\frac{4}{\epsilon^2}\Big(H_1+H_0\Big)+\frac{4}{\epsilon}\Big(-H_{1,1}-H_{0,1}+H_0+H_1-\zeta_2\Big)\\
&+4\Big(H_{1,1,1}+H_{0,1,1}-H_{1,1}-H_{0,1}+H_1\zeta_2+H_1+H_0\zeta_2+H_0-\zeta_3-\zeta_2\Big)\Big]\\
&+\Big[\frac{1}{\epsilon}\Big(-4H_1-4H_0+2z\Big)+\Big(4H_{1,1}+4H_{0,1}-2(2+z)H_1-4H_0+4\zeta_2-2+6z\Big)\Big]\Big\}\,,
\end{aligned}
$$

$$
\begin{aligned}
f^{VR}_{q/q}(z,\epsilon,\alpha,s)&=C_fC_a\Big\{\delta(1-z)\Big[\frac{4s}{\alpha\epsilon^3}-\frac{4s}{\alpha\epsilon}\zeta_2+\frac{8s}{3\alpha}\zeta_3-\frac{1+s}{\epsilon^4}+\frac{3(1+s)}{\epsilon^2}\zeta_2+\frac{28(1+s)}{3\epsilon}\zeta_3\\
&+\frac{57(1+s)}{4}\zeta_4\Big]+\tilde{p}_{qq}(z)\Big[\frac{2}{\epsilon^3}+\frac{2}{\epsilon}\Big(2H_{1,0}+\zeta_2\Big)+\Big(-8H_{1,1,0}-4H_{1,0,1}-4H_1\zeta_2+\frac{4}{3}\zeta_3\Big)\Big]\\
&+\Big[-\frac{2-2z}{\epsilon^2}+\frac{2z}{\epsilon}+\Big(-4(1-z)H_{1,0}-2zH_1-2(1-z)\zeta_2+2+2z\Big)\Big]\Big\}\\
&+C_f^2\Big\{\tilde{p}_{qq}(z)\Big[\frac{4}{\epsilon^2}H_0-\frac{4}{\epsilon}\Big(2H_{1,0}+H_{0,1}+2\zeta_2\Big)+\Big(16H_{1,1,0}+8H_{1,0,1}+4H_{0,1,1}+8H_1\zeta_2+4H_0\zeta_2-4\zeta_3\Big)\Big]\\
&+\Big[\frac{2}{\epsilon}\Big(-2(1-z)H_0-z\Big)+\Big(8(1-z)H_{1,0}+4(1-z)H_{0,1}+2H_1z+4(1-z)\zeta_2-2-2z\Big)\Big]\Big\}\,,
\end{aligned}
\tag{6.41}
$$

$$
f^{VR}_{\bar{q}/q}(z,\epsilon,\alpha,s)=f^{VR}_{q'/q}(z,\epsilon,\alpha,s)=0\,. \tag{6.42}
$$

All other VR contributions to $\mathcal{B}$ and $\bar{\mathcal{B}}$ are related by charge conjugation or flavor symmetry to these results, as described in section 4.7. The NNLO contribution to $\mathcal{B}'_{g/j}$ are not provide in this work.

Through the last argument $s$, which is chosen as $+1$ in the collinear and as $-1$ in anti-collinear region, respectively, the functions $f^{VR}_{a/b}$ differ slightly between these regions in case of the diagonal splittings ($gg$, $qq$). For the off-diagonal splittings up to $\alpha^0$, $\epsilon^0$ no difference arises.

The highest poles appearing above are $\epsilon^{-4}$ and $\alpha^{-1}$. The sum of both exponents is minimally $-4$ and appears for endpoint contributions in diagonal splittings in the color structure $C_a^2$ for the gluon and $C_aC_f$ for the quark, respectively. $\alpha^{-1}$ only appears in the diagonal contributions. This is just the pole structure, which is to expected bearing in mind that per external particle one integral can be regulated by $\alpha$ instead of $\epsilon$.

For each combination of partons, only the expected combinations out of two color factors appear. The dependence on the analytic regulator scale $v$ could be completely absorbed into $L_a$ for the anti-collinear and $L_c$ for the collinear region, respectively. Both of them are multiplied by the corresponding regulator $\alpha$.

The dependence on the transverse scale appeared solely in quadratic form and was contained in $L_\perp$ and for our convenience also partly in $L_c$. All three logarithms of mass ratios could completely be absorbed in the first factor of eqns. (6.36, 6.37). No further mass ratios appeared.

Before discussing the details of refactorization section 8.1, we will discuss the remaining

NNLO contribution form the double emission in the next section.

# 7 Double Emission

Now let us consider the real-real (RR) contribution to $\mathcal{B}_{a/b}$ and $\Phi_{a/b}$ in eqn. (4.14) with $n = 2$ and no virtual correction. This is the remaining NNLO contribution. Similarly to the simpler cases discussed in the last chapters, we will start from the relevant QCD like matrix element $m$, provide the relevant amplitude topologies and then consider the integration of $\overline{|m|^2}$. As the last step is very complicated, its discussion is split in several steps. We start with some basic reformulations and then specify various parametrizations of integration variables, which we used. Then we discuss relations among the various integrals and finally solve the relevant integrals.

## 7.1 Topologies

Let us call the emitted partons $j_1$ and $j_2$ with momenta $l$ and $k - l$, such that their sum is given by $k$. The incoming parton $b$ has momentum $p$. Following the considerations of sections 4.2 and 4.2.1, we identify the QCD like matrix element $m_{a/b,j_1,j_2}$, according to eqns. (4.5, 4.11, 4.12). This matrix element we consider at tree level.

The three amplitude topologies contained in $m_{a/b,j_1,j_2}$ are given in Figure 7.1. In the first one, the incoming parton emits first a parton of momentum $l$ and later one of momentum $k - l$. In the second one the order is crossed. In the third amplitude topology, a parton of momentum $k$ is emitted, which splits into the two final partons. For all three amplitude topologies one propagator carries momentum $p - k$. The other momentum is either $p - l$, $p - k + l$ or $k$ depending on the amplitude topology. By shrinking the propagator with the second momentum we receive the same amplitude subtopology from all of them.

As mentioned earlier, each propagator of momentum $h \in (p - k, p - l, p - k + l, k)$ can produce both, a power of $1/h^2$ and of $1/\bar{n} \cdot h$. Hence, the squared matrix element can contain both denominators up to second power for each of the propagators present. In each combination of diagrams $p - k$ is present twice. The other two propagators are any of $p - l$, $p - k + l$ or $k$, where the same propagator is taken twice if the diagram is combined with itself. Moreover, for external gluons single powers of the light-cone propagators of the corresponding momenta can arise.

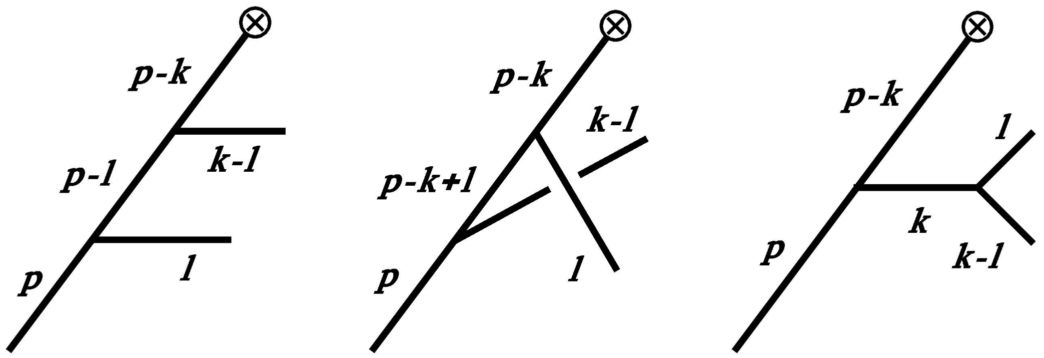



Figure 7.1: Amplitude topologies for real-real case.

## 7.2 Basic Expression

For the topologies of the squared matrix element $\overline{|M^{(2)}|^2}$, which is related to the matrix element $m$ discussed above by eqn. (4.13), we have to solve the following types of integrals

$$\left(\frac{\alpha_s}{4\pi}\right)^2 \left\{\mathcal{B}_{a/b}^{(RR)}\,;\,\phi_{a/b}^{(RR)}\right\}(z) = \tag{7.1}$$
$$\int \frac{\mathrm{d}^d k}{(2\pi)^d}\delta(k_- - z_- p_-)\left\{e^{ik_T\cdot x_T};1\right\}\int \frac{\mathrm{d}^d l}{(2\pi)^{d-2}}\delta^+(l^2)\delta^+([k-l]^2)\left(\frac{v^2}{l_+[k-l]_+}\right)^{\alpha}\overline{|M_{a/b}^{(2)}|^2}\,,$$

with the maximal powers of denominators as discussed above. Most terms in $\overline{|M^{(2)}|^2}$ will not have the maximal powers of the denominators, due to the presence of the numerator. As before, for the anti-collinear case we rename $p \sim n \leftrightarrow \bar{p} \sim \bar{n}$ and obtain the same expression but with the analytic regulator on the minus components of the momenta. For the gluon nTPDF, the contractions with the second projector gives the same expression, but with $\overline{|M'^{(2)}|^2}$. In this case, the numerator can also contain contractions of $x_\perp$ with any of the momenta.

The $k$ integral we again split via eqn. (5.4) to light-cone and transverse components. The $k_-$ integral is performed trivially by the relevant $\delta$-function such that $k_- = z_- p_-$. Introducing

$$y = k_T^2/(k_+ k_-) \tag{7.2}$$

and making use of the $\theta$-functions the $k_+$ integral can be written as

$$\int_{k_T^2/k_-}^{\infty} \mathrm{d}k_+ = \frac{k_T^2}{k_-}\int_0^1 \frac{\mathrm{d}y}{y^2}\,. \tag{7.3}$$

## 7.3 Parametrization

Using $k_T^2$, $y$ and $k_-$ to express $k^2$ and $-(p-k)^2$ we receive

$$
\begin{aligned}
k^2 &= k_+k_- - k_T^2 = k_T^2\left(\frac{1-y}{y}\right) ,\\
-(p-k)^2 &= (p_-k_+ - k^2) = \frac{k_T^2}{yz_-}[1-z_-(1-y)] .
\end{aligned}
\tag{7.4}
$$

For later convenience we define

$$
D_5 = y, \quad D_6 = 1-y, \quad D_7 = [1-z_-(1-y)] . \tag{7.5}
$$

The remaining two squared momenta $(p-l)^2$ and $(p-k+l)^2$, which can appear in the denominator, we rewrite in terms of light cone components

$$
(p-l)^2 = -p_-l_+ , \tag{7.6}
$$

$$
(p-k+l)^2 = -p_-[k-l]_+ , \tag{7.7}
$$

where we used $p^2, l^2, (k-l)^2 = 0$ and eqn. (4.29).

To calculate the $l$-integrals we boost to the rest frame of $k$, i.e. we parameterize $k$ as

$$
k^\mu = k_T\sqrt{\frac{1-y}{y}}(1,0,0,0)^\mu , \tag{7.8}
$$

where the prefactor is determined from eqn. (7.4). Having chosen this class of frames, we still have the freedom to rotate in $d-1$ dimensions, as $\vec{k}=0$.

Depending on the considered case we choose different parametrizations of the remaining vectors $n$, $\bar{n}$, $l$ and if present $x_\perp$. A parametrization A for which $\bar{n}^\mu$ has a simple form and a parametrization C for which $n^\mu$ has a simple form. The parametrization A is our default choice. It is applied if any of the denominators $\bar{n}\cdot(p-l)$ or $\bar{n}\cdot(p-k+l)$ is present or if the analytic regulators are located at $\bar{n}\cdot l$ or $\bar{n}\cdot(k-l)$, as occurs for the **a**nti-collinear case. In the other cases, including many integrals appearing for the **c**ollinear case, we use the parametrization C. Let us now provide details to these parametrizations.

### Parametrization A

For parametrization A, we rotate to the frame where

$$
\begin{aligned}
\bar{n}^\mu &= c_{\bar{n}}(1,0,0,z_0)^\mu,\\
n^\mu &= c_n(1,0,y_2,y_3)^\mu,\\
x_\perp^\mu &= (x_0,x_1,x_2,x_3)^\mu,\\
l^\mu &= c_l(r_0,\sin\theta_1\sin\theta_2\cos\theta_3,\sin\theta_1\cos\theta_2,\cos\theta_1)^\mu.
\end{aligned}
\tag{7.9}
$$

This is we set as many components of the vectors $\bar{n}$, $n$ and $x_\perp$ to 0 as possible. For all vectors we suppressed the $d-4$ component, which is chosen to vanish for all vectors but $l$. The only trace of it is the presence of $\cos\theta_3$ in $l_1$.

While the angles $\theta_1$, $\theta_2$ and $\theta_3$ are integration variables for the $l$ integral, all other parameters appearing above will be expressed in terms $y$, $k_-$ as well as vector products between $k_\perp$ and $x_\perp$. To this end we use the following constraints on the scalar products

$$\begin{aligned} n\cdot n &= 0\,, & \bar{n}\cdot\bar{n} &= 0\,, & l\cdot l &= 0\,, \\ n\cdot x_\perp &= 0\,, & \bar{n}\cdot x_\perp &= 0\,, & k\cdot x_\perp &= k_\perp\cdot x_\perp\,, \\ n\cdot\bar{n} &= 2\,, & \bar{n}\cdot k &= k_-\,, & n\cdot k &= k_+\,. \end{aligned} \tag{7.10}$$

and the fact that in its rest frame $k$ decays into two massless particles of same energy, implying $l_0 = \frac{k_0}{2}$.
Then we find for parametrization A

$$\bar{n}^\mu = \frac{k_-}{k_T}\sqrt{\frac{y}{1-y}}(1,0,0,1)^\mu\,, \tag{7.11}$$

$$n^\mu = \frac{k_T}{k_-}\frac{1}{\sqrt{y(1-y)}}(1,0,-2\sqrt{y(1-y)},2y-1)^\mu\,, \tag{7.12}$$

$$x_\perp^\mu = \frac{1}{k_T}\left(k\cdot x_\perp\sqrt{\frac{y}{1-y}},\ \sqrt{k_T^2x_T^2-(k\cdot x_\perp)^2},\ -k\cdot x_\perp,\ k\cdot x_\perp\sqrt{\frac{y}{1-y}}\ \right)^\mu\,, \tag{7.13}$$

$$l^\mu = \frac{k_T}{2}\sqrt{\frac{1-y}{y}}(1,\sin\theta_1\sin\theta_2\cos\theta_3,\sin\theta_1\cos\theta_2,\cos\theta_1)^\mu\,. \tag{7.14}$$

This parametrization leads to the following representation of the scalar products

$$\begin{aligned} &\bar{n}\cdot l = k_-D_1\ , \quad \bar{n}\cdot(k-l) = k_-D_3\ , \\ &n\cdot l = \frac{k_T^2}{k_-y}D_2\ , \quad n\cdot(k-l) = \frac{k_T^2}{k_-y}D_4\ , \\ &\bar{n}\cdot(p-l) = p_-D_8\ , \quad \bar{n}\cdot(p-k+l) = p_-D_9\ , \end{aligned} \tag{7.15}$$

where the denominators $D_1, D_2, D_3, D_4, D_8$ and $D_9$ are defined as

$$\begin{aligned} D_1 &= \frac{1-\cos\theta_1}{2} \\ D_3 &= 1-D_1 = \frac{1+\cos\theta_1}{2} \\ D_2 &= \frac{1}{2}\left(1+2\sqrt{y(1-y)}\sin\theta_1\cos\theta_2-(2y-1)\cos\theta_1\right) \\ D_4 &= 1-D_2 = \frac{1}{2}\left(1-2\sqrt{y(1-y)}\sin\theta_1\cos\theta_2+(2y-1)\cos\theta_1\right) \end{aligned} \tag{7.16}$$

$$D_8 = 1 - z_- D_1 = 1 - z_- \frac{1-\cos\theta_1}{2}$$
$$D_9 = (1+z) - D_8 = 1 - z_- D_3 = 1 - z_- \frac{1+\cos\theta_1}{2} \,.$$

The parametrizations of $k^2$ and $-(p-k)^2$ have been given in (7.4). Products of momenta with $x_\perp$ only appear in the numerator. Hence, we can map them to $x_\perp \cdot l$ and $x_\perp \cdot k$. While $x_\perp \cdot k$ will be kept, we rewrite

$$x_\perp \cdot l = \frac{x_\perp \cdot k}{2}\Big(\frac{D_1 - D_4}{y} + 1\Big) - x_1 l_1 \,, \tag{7.17}$$

with

$$x_1 l_1 = \frac{1}{2}\sqrt{\frac{1-y}{y}}\sqrt{x_T^2 k_T^2 - (k\cdot x_\perp)^2}\,\sin\theta_1 \sin\theta_2 \cos\theta_3 \,, \tag{7.18}$$

and expand this.

#### Parametrization C

We now discuss the parametrization C, where $n^\mu$ is chosen in the most simple way. The considerations correspond to the ones of the last subsection, we only interchange the form of $n$ and $\bar{n}$ in eqn. (7.9). This implies

$$n^\mu = \frac{k_T}{k_-}\frac{1}{\sqrt{y(1-y)}}(1,0,0,1)^\mu \,, \tag{7.19}$$
$$\bar{n}^\mu = \frac{k_-}{k_T}\sqrt{\frac{y}{1-y}}(1,0,-2\sqrt{y(1-y)},2y-1)^\mu \,, \tag{7.20}$$

while the form of $l^\mu$ and $x_\perp^\mu$ is the same as in eqns. (7.13, 7.14). This leads to the following representation of the scalar products

$$\begin{aligned} n\cdot l &= \frac{k_T^2}{k_- y} D_1 \,, \quad n\cdot(k-l) = \frac{k_T^2}{k_- y} D_3 \,, \\ \bar{n}\cdot l &= k_- D_2 \,, \quad \bar{n}\cdot(k-l) = k_- D_4 \,, \end{aligned} \tag{7.21}$$

with $D_i$ defined in eqn. (7.16). Note that we do not provide explicit expressions for $\tilde{D}_8$ and $\tilde{D}_9$ in this parametrization, as it will not be used in presence of those denominators.

## 7.4 Basic Integration

Using the explicit parametrizations discussed in the last section, we can rewrite the $l$ integral in eqn. (7.1) as follows

$$\begin{aligned}\int \mathrm{d}^d l\, \delta^+((k-l)^2)\delta^+(l^2) &= \int \mathrm{d}l_0 \frac{1}{2k_0}\delta(l_0 - k_0/2) \int \mathrm{d}|\vec{l}|\, |\vec{l}|^{d-2} \frac{1}{2|\vec{l}|}\delta(|\vec{l}| - l_0) \int \mathrm{d}\Omega_{d-1} \\ &= 2^{-3+2\epsilon} k_0^{-2\epsilon} \int \mathrm{d}\theta_1 \sin^{1-2\epsilon}\theta_1 \int \mathrm{d}\theta_2 \sin^{-2\epsilon}\theta_2 \int \mathrm{d}\Omega_{d-3}\ . \end{aligned} \tag{7.22}$$

$k_0^2$ can be rewritten as $k_0^2 = k^2 = k_T^2 \frac{1-y}{y}$. The result for $\int \mathrm{d}\Omega_{d-3}$ depends on the power $n$ of $\cos^n \theta_3$ present. Those factor only enters via eqn. (7.18). We obtain $n = 2$ for $(x_1 l_1)^2$, $n = 1$ for $(x_1 l_1)^1$ and $n = 0$ if this factor is not present, which is obviously the most common case. Using eqns. (5.14, B.3) and performing the $\theta_3$ integral, we receive

$$\int \mathrm{d}\Omega_{d-3} = \int_0^\pi \mathrm{d}\theta_3 \sin^{-1-2\epsilon}\theta_3\, \cos^n\theta_3 \int \mathrm{d}\Omega_{d-4} = \begin{cases} 2^{1-2\epsilon}\pi^{-\epsilon}\Gamma(1-\epsilon)/\Gamma(1-2\epsilon)\ , & n=0 \\ 0 & , \ n=1 \\ 2^{1-2\epsilon}\pi^{-\epsilon}\Gamma(1-\epsilon)/\Gamma(2-2\epsilon)\ , & n=2 \end{cases} \tag{7.23}$$

This is, terms with $n = 1$ do not contribute. The terms with $n = 2$ can be related to those with $n = 0$: The result for $\int \mathrm{d}\Omega_{d-3} \cos^n \theta_3$ just differs by the factor $1/(1-2\epsilon)$, as follows from the last equation with eqn. (B.2). We write

$$\frac{(l_1 x_1)^2}{\cos^2\theta_3} = \frac{1}{2^2}\frac{1-y}{y}(k_T^2 x_T^2 - (k\cdot x_\perp)^2)((1-\cos^2\theta_1) - (\sin\theta_1\cos\theta_2)^2)\ , \tag{7.24}$$

where $\sin\theta_1 \cos\theta_2$ and $\cos\theta_1$ can be expressed via $D_4$ and $D_1$ and the whole expression can be expanded. Hence, in this sense, the individual terms with $n = 2$ are of the same structure as the terms with $n = 0$.

Combining the integral of eqn. (7.22) with the one over $y$ and removing a factor independent of $y$, $\theta_1$ and $\theta_2$, we define the following integrals

$$\begin{aligned} I^{\mathrm{RR}}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) = &\int_0^1 \mathrm{d}y \frac{1}{2\pi}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \int_0^\pi \mathrm{d}\theta_1 \sin^{1-2\epsilon}\theta_1 \int_0^\pi \mathrm{d}\theta_2 \sin^{-2\epsilon}\theta_2 \\ &\times \prod_{i=1}^{9} D_i^{-a_i} \end{aligned} \tag{7.25}$$

where we used $D_i$ as defined in (7.5, 7.16), which depend on $\theta_1$, $\theta_2$, $y$ and, in case of $D_7$, $D_8$ and $D_9$, also $z$. Note the inclusion of an additional factor of $\Gamma(1-\epsilon)$ w.r.t. eqn. (7.23) to avoid occurrences of $\gamma_e$ in the expansion of the integrals.

| Index | Momentum | LC/Sq |
|---|---|---|
| $a_9$ | $p-k+l$ | LC |
| $a_8$ | $p-l$ | LC |
| $a_7$ | $p-k$ | Sq |
| $a_6$ | $k$ | Sq |
| $a_5$ | - | - |
| $a_4$ | $p-k+l$ | Sq |
| $a_2$ | $p-l$ | Sq |
| $a_3$ | $k-l$ | LC |
| $a_1$ | $l$ | LC |

Table 7.1: Denominators as source of positive values of the indices in parameterization A. For momentum $h$ the mark Sq refers to the presence of $1/h^2$ and the mark LC to the presence of $1/(\bar{n}\cdot h)$.

### 7.4.1 Indices and Topologies

From the discussion of topologies and parametrizations in section 7.1 and 7.3 respectively, the maximal values of indices and their combination follows. To recall the relations and provide an overview, we collect in Table 7.1 which index is increased by the presence of which denominator. Reading the table from the right to the left, one can also easily obtain the maximal values of indices for a given combination of two diagrams. The relation between indices and denominators depends on the parameterization. The table refers to parameterization A. If $a_9, a_8 = 0$ the indices of the alternative parameterization C directly follow by exchanging $(a_1, a_2, a_3, a_4) \leftrightarrow (a_2, a_1, a_4, a_3)$. The momenta in that table refer to the momenta of the propagators in the diagram of the topologies. Besides $1/h^2$, the propagator $h$ can produce $1/(\bar{n}\cdot h)$ if it refers to a gluon. Such a term can also appear for the external momenta if the corresponding parton is a gluon.

From the topologies it is clear that the maximal values of $a_1$ and $a_3$ is 1, while it is 2 for all other indices but $a_5$ (parameterization A). Moreover we observe $a_9 + a_8 + a_6 \leq 2$, $a_2 + a_4 + a_6 \leq 2$. Where we only referred to the integer part of the indices.

The index $a_5$ enters in a different way than the other indices: On the one hand we received $y^{-2}$ from the integration measure and also $l\cdot x_\perp$ contains $y^{-1}$. On the other hand each negative power of a squared momenta $1/h^2$ with $h \in \{k, p-k, p-l, p-k-l\}$ produces $y^{+1}$. In the end we never had to deal with values of $a_5$ higher than 1 and the value 1 only appeared in the presence of $l\cdot x_\perp$.

Some of the indices are shifted from the integers by the dimensional regulator $\epsilon$ or the analytic regulator $\alpha$. These shifts originate from the presence of $k_0^{-2\epsilon} = (k_T^2)^{-\epsilon} D_5^{\epsilon} D_6^{-\epsilon}$ in eqn. (7.22) and the analytic regulator $\alpha$ on $n\cdot l$ and $n\cdot(k-l)$ in eqn. (7.1). Therefore, the indices $a_7$, $a_8$ and $a_9$ are integers; $\epsilon$ is contained in $a_5$ and $a_6$ and $\alpha$ in either $a_1$ and $a_3$ or $a_2$ and $a_4$. For the collinear case $\alpha$ is also contained in $a_5$. With the proper choice of parametrization, the mappings discussed in section 7.6 and the fact that in many cases

| Index | $a_1$ | $a_3$ | $a_5$ | $a_6$ |
|---|---|---|---|---|
| Regulator part | $\alpha$ | $\alpha$ | $-\epsilon - m\alpha$ | $\epsilon$ |

Table 7.2: Indices containing regulators.

$\alpha$ can be dropped, we will always be able to have integer powers $a_2$ and $a_4$. This is very useful, because $D_2$ and $D_4$ have a rather complicated form.

Then the only non-integer indices are $a_1$, $a_3$, $a_5$ and $a_6$. They are shifted from the integers as given by Table 7.2, where $m = 0$ in the anti-collinear case whereas $m = 2$ in the collinear case.

Solving the integrals $I^{\mathrm{RR}}$ will be the main concern of the following subsections. But before discussing $I^{\mathrm{RR}}$ itself, let us discuss the factors multiplying this integral in the next subsection. Among those especially the integrations over the remaining components of $k$.

## 7.5 Factorized Integrals and Their Poles

A very useful feature of our parametrizations is that the dependence on $k_T$ is completely factorized from the denominators $D_1$ to $D_9$ and as we have seen around eqn. (7.22) also from the $l$ integral. Then however, the **$\mathbf{k_T}$-integral** in eqn. (7.1) is of the same form as at NLO. This means, perturbative corrections to $\phi$ vanish, because the integral is scaleless, and for $\mathcal{B}$, the $k_T$ integral can be solved by eqns. (5.13, 5.16). This covers all relevant cases, as follows from the discussion around eqns. (7.23, 7.24). The powers of $k_T^2$ we encounter are

$$w = -1 - \epsilon - m\alpha \quad \text{or} \quad b = -2 - \epsilon - m\alpha \quad \text{with } m = 0, 2\,. \tag{7.26}$$

Then the $\Gamma$-function in the numerator contains a pole in one of the regulators. To reveal this pole, one can map each $\Gamma$-function to $\Gamma(1 + \text{regulators})$. Besides the finite factors, we then find the factor

$$\frac{1}{-2\epsilon - m\alpha} = -\frac{1}{2} \cdot \frac{1}{\epsilon + \alpha m/2}\,. \tag{7.27}$$

For $m = 0$ it is obvious that this is a simple pole in $\epsilon$. For $m = 2$ we need to recall that we have to expand first in $\alpha$ then in $\epsilon$. Thus we receive in that case

$$\frac{1}{\epsilon + \alpha} = \frac{1}{\epsilon} \cdot \frac{1}{1 + \alpha/\epsilon} = \frac{1}{\epsilon} - \frac{\alpha}{\epsilon^2} + \frac{\alpha^2}{\epsilon^3} + \mathcal{O}(\alpha^3)\,, \tag{7.28}$$

which is free of poles in $\alpha$ but contains poles in $\epsilon$. More precisely a single pole multiplied by powers of $\frac{\alpha}{\epsilon}$. To take account of this structure, we will expand the $I^{\mathrm{RR}}$ integrals in terms of $\tilde{\alpha} = \frac{\alpha}{\epsilon}$ instead of $\alpha$.

As discussed earlier, the **k₋-integral** in eqn. (7.1) is performed by the $\delta$-function and $k_-$ will be replaced by $p_- z_-$. The lowest power of $z_-$ we receive in this way is $-1 \pm 2\alpha$. Using the plus-distribution as explained around eqn. (B.32), we write for these terms

$$z_-^{-1\pm 2\alpha} = \pm\frac{1}{2\alpha}\delta(z_-) + \left(z_-^{-1\pm 2\alpha}\right)_+ . \tag{7.29}$$

The first term contains a pole in $\alpha$ together with $\delta(z_-)$.

Taken together the factor in front of $I^{\mathrm{RR}}$ can contain poles up to $(\alpha/\epsilon)^{-1}\epsilon^{-2}$ if $\delta(z_-)$ appears or $\epsilon^{-1}$ if it does not.

### Required Order of Expansion

The required order of expansion for the $I^{\mathrm{RR}}$ integrals is then the inverse of the poles in the factor just discussed. This implies the integrals $I^{\mathrm{RR}}$ in eqn. (7.25) have to be expanded to $(\alpha/\epsilon)^1$, $\epsilon^2$ in presence of $\delta(z_-)$ and to $(\alpha/\epsilon)^0$, $\epsilon^1$ in other cases.

There is a special case, where the expansion is needed only to $(\alpha/\epsilon)^0$, $\epsilon^0$. This is for the integrals which appear solely for the second gluon tensor structure. They are introduced by the terms proportional to $y^{-2}$ in the expansion of $l \cdot x_\perp$, which comes with $-x_\perp^2 k_\perp^2 + (x_\perp \cdot k_\perp)^2(d-2)$. Combined with the common power of $k_T^2$ their $k_T$ integration does not produce the factor $\frac{1}{\epsilon + m\alpha/2}$ discussed around eqn. (7.27), but is free of any poles including terms of the from $(\alpha/\epsilon)^n$.

## 7.6 Relations among Integrals

Let us now consider the real-real integrals of eqn. (7.25) as functions of the indices $a_1, \ldots, a_9$; $\epsilon$ and the momentum fraction $z$. The maximal values of the indices have been discussed in section 7.4. Of course, in addition to the denominators from the propagators there usually is a numerator which lowers the values of some the indices w.r.t. their maximal value. For some indices the value can become as low as $-4$. Of course, negative indices are never a problem, as they correspond to numerators instead of denominators and can easily be expanded, also there is quite some freedom how to choose them. In fact, in section 7.7.4 we will in some cases push some indices to negative values to receive simpler integrals. It is apparent that we will not choose $a_{7,8,9} < 0$, however.

Considering the whole set of integrals with indices in this range, we are going to find linear dependencies between them. Those we will use to reduce the number and/or the complexity of the integrals we have to calculate.

Due to the presence of the analytic regulators, the $\delta$-functions and the (not Lorentz invariant) function $e^{ik_T \cdot x_T}$, we were not able to use the powerful method of integration by parts in a profitable way to reduce the integrals to a small set of master integrals. Hence, we applied other, basic techniques to reduce the number of integrals and their complexity. The main method is partial fraction decomposition, another relabeling invariance. Since are only interested in the finite term in $\alpha$, we can drop this regulator for some integrals.

| $(i,j)$ | (1,3) | (2,4) | (1,8) | (1,9) | (3,8) | (3,9) | (8,9) | (5,6) | (6,7) | (5,7) |
|---|---|---|---|---|---|---|---|---|---|---|
| $b_{ij}$ | 1 | 1 | 1 | $1/z$ | $1/z$ | 1 | $\frac{1}{1+z}$ | 1 | 1 | $1/z$ |
| $c_{ij}$ | 1 | 1 | $z_-$ | $\frac{-z_-}{z}$ | $\frac{-z_-}{z}$ | $z_-$ | $\frac{1}{1+z}$ | 1 | $z_-$ | $\frac{-z_-}{z}$ |

Table 7.3: Partial fraction identities between integrals.

| $(i,j)$ | $(z_-,z)$ | $(z_-,z_+)$ | $(z,z_+)$ |
|---|---|---|---|
| $b_{ij}$ | 1 | 1/2 | 1 |
| $c_{ij}$ | 1 | 1/2 | -1 |

Table 7.4: Partial fraction identities for $z$, $z_- = 1 - z$ and $z_+ = 1 + z$.

Then we are more flexible in choosing parametrization. While the first two methods lead to exact identities, the last one essentially corresponds to an expansion in $\alpha$.

We will discuss the three methods in the following subsections. After having pointed out the relations they imply, we will discuss the tactic we used to reduce our expression to a small and simplified set of integrals. The results for the non trivial integrals will then be listed in section D.

### 7.6.1 Partial Fraction Decomposition

A basic but very generic method is partial fraction decomposition (PF). It is discussed in section A. Here, we apply this method for the denominators $D_i$ introduced in eqn. (7.16). The relevant relations for each pair $D_i$ and $D_j$ are eqns. (A.7, A.8, A.9), where the function $I$ corresponds to $I^{\mathrm{RR}}$ with the indices $a_k$ suppressed for $k \neq i, j$. In the notation of these equations Table 7.3 provides the relevant parameters. The coefficients can be obtained most easily by considering eqn. (A.3) with eqns. (7.5, 7.15). As allover this notes, we used $z_- = 1 - z$. PF of other combinations of denominators in the integral are not applied at this stage, as the related parameters would be complicated functions of the variables.

Despite the denominators in the integrals, also the prefactors containing $z$, $z_- = 1 - z$ and $z_+ = 1 + z$ should be partial fractioned. With the notation as above, the relevant parameters are given in Table 7.4.

### 7.6.2 Choosing Parametrization

As discussed in the section about parametrization, parametrization A is our default choice. In several cases it is, however, preferable to change to parametrization C. Such a change is feasible if $a_8 = a_9 = 0$. From eqns. (7.15, 7.21) and the definition (7.25) we obtain the relation

$$I^{\mathrm{RR}}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, 0, 0) = I^{\mathrm{RR}}(a_2, a_1, a_4, a_3, a_5, a_6, a_7, 0, 0)\ , \tag{7.30}$$

for changing between the parametrization A and C. This corresponds to an exchange the two pairs of indices $a_1 \leftrightarrow a_2$ and $a_3 \leftrightarrow a_4$.

### 7.6.3 Relabeling

As the integrals where obtained by rewriting the integrals over momentum, the freedom to relabel $l \leftrightarrow k - l$, where $k$ is unchanged, implies an additional relation for the integrals, which is

$$I^{\mathrm{RR}}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) = I^{\mathrm{RR}}(a_3, a_4, a_1, a_2, a_5, a_6, a_7, a_9, a_8)\ . \tag{7.31}$$

Note that the analytic regulators enter in a symmetric way between $l$ and $k - l$ and this relation is therefore not disturbed by their presence.

### 7.6.4 Tactics

As we have seen in the last subsections, there is quite a number of relations among the integrals. To arrive at a minimal or optimal set of integrals, we have to use them in a systematic way.

We start from integrals in our default parametrization A (see eqn. (7.15)), for which we provided an explicit form of $D_8$ and $D_9$.

In a first step, we use potential $\delta(1-z)$ functions multiplying the integral to remove all $z$ dependent denominators from that integral. This will set $a_7,\ a_8,\ a_9 = 0$ for the corresponding integrals. Note that a prefactor containing $\alpha^{-1}$ is always accompanied by such a $\delta$-function, as was pointed out around eqn. (7.29).

In the next step, we apply partial fraction decomposition by using eqn. (A.8) and the two related equations recursively in the way explained there for all possible combinations of indices as listed in Table 7.3, to remove or lower as many positive indices as possible. I.e. we consider the separate sets $(a_1, a_3, a_8, a_9)$, $(a_2, a_4)$ and $(a_5, a_6, a_7)$ to arrive at integrals, where at most one of the indices in each group is different from 0. Here and elsewhere in this subsection a statement like this is always meant up to regulators. At this stage the priority is to remove $a_9$, $a_8$ and $a_7$ if possible.

If the remaining index of a group is negative, we have the freedom to move it within the group. We choose here to only have negative occurrences of $a_1$, $a_2$ and $a_5$, but of none of the others. In the specific identities below, it is always understood if not stated otherwise that after their application we map potential negative indices $a_3$ or $a_4$ to $a_1$ and $a_2$.

This ensures for example, that integrals with $a_8$ or $a_9 > 0$ have zero indices for $a_1$ and $a_3$ as well as at least for one of $a_2$ or $a_4$. We will see in our discussion in the beginning of section 7.7.2 that integrals of this kind never have poles in $\alpha$. Because we are only interested in the expansion of the operator to $\alpha^0$, we drop $\alpha$ for them. Also for many integrals with $a_8, a_9 = 0$ that regulator can be dropped, as discussed in detail in the beginning of section 7.7.4. Whenever allowed, we drop $\alpha$.

For all integrals where we have to keep $\alpha$, we use eqn. (7.30) to ensure that the analytic regulators are always located at $a_1$ and $a_3$.

Using the eqn. (7.31) for integrals with $a_4 > 0$, we map $a_4$ to 0. If $a_3$ but none of $a_2$ and $a_4$ is positive, we use it to map $a_3$ to 0. If $a_9$ but not $a_2$ is positive, we use it to map $a_9$ to 0.

Having $a_4 = 0$, $D_2$ is the most complicated denominator present. Hence, an integral becomes less complicated if positive occurrences of $a_2$ can be avoided and if $a_2$ is as close to 0 as possible.

In terms where $\alpha$ can be set to zero and $a_8, a_9 = 0$, we can use eqn. (7.30) to get rid of some more integrals or reduce their complexity. Given the constraints above are fulfilled we use the relation in the following three cases: If $a_2 > 0$, $a_2 > a_1$ and $a_3 \leq 0$; If $a_2 < a_1 \leq 0$ and $a_3 \leq 0$; If $a_2 > a_3 > 0$ and $a_1 < 0$, we combine the equation with equation (7.31) to map

$$I^{\mathrm{RR}}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, 0, 0) = I^{\mathrm{RR}}(a_4, a_3, a_2, a_1, a_5, a_6, a_7, 0, 0) \ . \tag{7.32}$$

In all three cases integrals with simpler values of $a_2$ are obtained.

In addition to that, we can combine eqn. (7.31) and PF to receive further simplifications. Mostly this can be applied for negative values of $a_1$ or $a_2$.[1] Let us consider such integrals with arbitrary $a_5, a_6, a_7$ which have after PF vanishing $a_3, a_4, a_8, a_9$.

The first case we consider is if $a_1 < 0$ and $a_2 = 0$.

$$\begin{aligned} &I^{\mathrm{RR}}(-1, 0, 0, 0, a_5, a_6, a_7, 0, 0) = \frac{1}{2} I^{\mathrm{RR}}(0, \%) \text{ and} \\ &I^{\mathrm{RR}}(-3, \%) = -\frac{1}{4} I^{\mathrm{RR}}(0, \%) + \frac{3}{2} I^{\mathrm{RR}}(-2, \%) \ , \end{aligned} \tag{7.33}$$

where on the RHS and in the second line we suppressed the constant indices. They follow from combining eqn. (A.9) and eqn. (7.31).

In a similar way, we can derive the following identity

$$I^{\mathrm{RR}}(-2, -1, 0, 0, a_5, a_6, a_7, 0, 0) = \frac{1}{2} I^{\mathrm{RR}}(-2, 0, \%) + I^{\mathrm{RR}}(-1, -1, \%) - \frac{1}{4} I^{\mathrm{RR}}(0, 0, \%) \ , \tag{7.34}$$

where we suppressed the unchanged indices on the RHS. Relations corresponding to the last three also hold for $a_1 \leftrightarrow a_2$. Moreover, those equations also hold in the presence of the regulators.

Besides reducing the number of integrals with distinct indices $a_i$, the relations discussed in this section also reduced the complexity of individual integrals.

Relevant to note for the next section, in which we discuss the calculation of these integrals, is that $a_2$ and $a_4$ are free of regulators and $a_4 = 0$. Moreover, no integral will have $a_8$ and $a_9 \neq 0$ at once.

[1] Keep in mind that we here always move negative indices to $a_1$ and $a_2$.

## 7.7 Steps towards Solving Subclasses of Integrals

After having used relations among integrals as explained in section 7.6.4, we are left with several subclasses of integrals for which certain indices vanish up to regulators. In this section we will discuss the actual calculation of those. In many cases we will focus on the integrals appearing.

Due to the tactics of section 7.6.4, at most one index is positive in each of the groups of indices $\{a_1, a_3, a_8, a_9\}$, $\{a_2, a_4\}$ and $\{a_5, a_6, a_7\}$, while the others can be chosen to vanish up to regulators. The regulators appear as given in Table 7.2 and we choose $a_4 = 0$. In a first step we will rewrite the integrals in a more applicable form which also reveals the pole structure in a more obvious way. This will be used to find adjustments to our approach of section 7.6.4 to reduce the leftover integrals to simpler ones. Some subclasses of integrals will be solved in closed form, for some other classes the explicit algorithm for their calculation will be described, while for the remaining integrals we will sketch the calculation and provide the explicit results in Appendix D.
We will first discuss integrals with $a_{8,9} = 0$. Then we will turn to integrals with $a_8 + a_9 > 0$.

### 7.7.1 Integrals with $\mathbf{a_8, a_9 = 0}$

The first set of integrals, we consider, are those with $a_{8,9} = 0$. As the denominators connected with $a_5$ to $a_7$ do not depend on the angles we will split them up for some intermediate steps writing

$$
\begin{aligned}
&I^{\mathrm{RR}}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, 0, 0) = I_2^{\mathrm{RR}}(a_5, a_6, a_7) \times I_1^{\mathrm{RR}}(a_1, a_2, a_3, a_4) \ , \ \text{with} \\
&I_1^{\mathrm{RR}}(a_1, a_2, a_3, a_4) = \frac{1}{2\pi} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \int_0^\pi \mathrm{d}\theta_1 \sin^{1-2\epsilon}\theta_1 \int_0^\pi \mathrm{d}\theta_2 \sin^{-2\epsilon}\theta_2 D_1^{-a_1} D_2^{-a_2} D_3^{-a_3} D_4^{-a_4} \\
&I_2^{\mathrm{RR}}(a_5, a_6, a_7) = \int_0^1 \mathrm{d}y \, y^{-a_5} (1-y)^{-a_6} [1 - z_-(1-y)]^{-a_7} \ ,
\end{aligned}
\tag{7.35}
$$

note that in general $I_1^{\mathrm{RR}}$ is also a function of $y$ and the equation above means to first solve $I_1^{\mathrm{RR}}$ and plug in the result to perform the integration over $y$.
For the generic case with $a_4 = 0$ we rewrite

$$
\begin{aligned}
I_1^{\mathrm{RR}}(a_1, a_2, a_3, 0) =& \frac{2^{-4\epsilon}}{\pi} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v \, u^{-a_3-\epsilon} (1-u)^{-a_1-\epsilon} v^{-1/2-\epsilon} (1-v)^{-1/2-\epsilon} \\
& \times \left[ \left( \sqrt{u(1-y)} - \sqrt{y(1-u)} \right)^2 + 4v\sqrt{u(1-u)y(1-y)} \right]^{-a_2} ,
\end{aligned}
\tag{7.36}
$$

by the change of variables to

$$
u = \frac{1+\cos\theta_1}{2} \ , \qquad v = \frac{1+\cos\theta_2}{2} \ . \tag{7.37}
$$

But before discussing the generic case in detail, let us discuss the simple cases.

### Simple Integrals

For certain subclasses $I_1^{\mathrm{RR}}$ can be solved in a closed form straightforwardly. This is when two of the four indices vanish exactly. As discussed above, we always choose $a_4 = 0$. For those integrals where in addition $a_2 = 0$, we receive

$$I_1^{\mathrm{RR}}(a_1, 0, a_3, 0) = \frac{\Gamma(1-\epsilon-a_1)\Gamma(1-\epsilon-a_3)}{\Gamma(2-2\epsilon-a_1-a_3)} \quad , \tag{7.38}$$

as can be seen from eqn. (7.36) and (B.26). This implies via eqns. (7.35) and (B.25)

$$\begin{aligned} I^{\mathrm{RR}}(a_1, 0, a_3, 0, a_5, a_6, a_7, 0, 0) =& \frac{\Gamma(1-\epsilon-a_1)\Gamma(1-\epsilon-a_3)}{\Gamma(2-2\epsilon-a_1-a_3)} \frac{\Gamma(1-a_5)\Gamma(1-a_6)}{\Gamma(2-a_5-a_6)} \\ & \times {}_2F_1(1-a_6, a_7, 2-a_5-a_6, z_-) \, , \end{aligned} \tag{7.39}$$

where we find a Hypergeometric function. Properties of such functions are discussed in section B.3.

In most cases, we will have $a_2 \neq 0$ and the equation above cannot be applied. Although we can reduce the integer part of $a_1$ or $a_3$ to zero, in general those indices are regulated by $\alpha$ and therefore do not vanish exactly. Fortunately, most integrals only need to be determined to order $\alpha^0$ and do not contain a poles in this regulator. For those integrals we can drop $\alpha$ and will be able to choose $a_1$ or $a_3$ to vanish. Hence, those subclasses are quite relevant. For their integrals we find

$$I_1^{\mathrm{RR}}(a_1, a_2, 0, 0) = \frac{\Gamma(1-\epsilon-a_1)\Gamma(1-\epsilon-a_2)}{\Gamma(2-2\epsilon-a_1-a_2)} \; {}_2F_1(a_1, a_2; 1-\epsilon; y) \, , \tag{7.40}$$

$$I_1^{\mathrm{RR}}(0, a_2, a_3, 0) = \frac{\Gamma(1-\epsilon-a_2)\Gamma(1-\epsilon-a_3)}{\Gamma(2-2\epsilon-a_2-a_3)} \; {}_2F_1(a_2, a_3; 1-\epsilon; 1-y), \tag{7.41}$$

If $a_7 = 0$, we can straightforwardly carry out the remaining $y$ integral by means of eqn. (B.22). This results in

$$I^{\mathrm{RR}}(a_1, a_2, 0, 0, a_5, a_6, 0, 0, 0) = \tag{7.42}$$
$$\frac{\Gamma(1-a_5)\Gamma(1-a_6)\Gamma(1-a_1-\epsilon)\Gamma(1-a_2-\epsilon)}{\Gamma(2-a_5-a_6)\Gamma(2-a_1-a_2-2\epsilon)} {}_3F_2\left( \begin{matrix} a_1, a_2, 1-a_5 \\ 2-a_5-a_6, 1-\epsilon \end{matrix} ; 1 \right) ,$$

$$I^{\mathrm{RR}}(0, a_2, a_3, 0, a_5, a_6, 0, 0, 0) = \tag{7.43}$$
$$\frac{\Gamma(1-a_5)\Gamma(1-a_6)\Gamma(1-a_2-\epsilon)\Gamma(1-a_3-\epsilon)}{\Gamma(2-a_5-a_6)\Gamma(2-a_2-a_3-2\epsilon)} {}_3F_2\left( \begin{matrix} a_2, a_3, 1-a_6 \\ 2-a_5-a_6, 1-\epsilon \end{matrix} ; 1 \right) .$$

In all of the last equations we have not stated the allowed values of indices. We assumed that the generic values of the non vanishing $\{a_i\}$ lie in the allowed range. The final equation we analytically continue to other relevant values if necessary. This is of course only sensible, if the obtained result is finite. The non-integer part of the $\{a_i\}$ are as stated in Table 7.2, for the last two cases we however set $\alpha = 0$. With these values all but eqn. (7.43) are finite. The later is **only finite if either $\mathbf{a_2 \leq 0}$, $\mathbf{a_3 \leq 0}$ or $\mathbf{a_2 + a_3 + a_5 - 1|_{\epsilon=0} \leq 0}$. Otherwise** it is not applicable and it was **not allowed to drop $\alpha$.** In those cases, the $y$ integral performed in the last step has an unregulated pole at 0, as can best be seen with the help of eqn. (B.19).

There are also some integrals with $a_2, a_7 \neq 0$ which can be calculated without much effort from eqns. (7.40, 7.41). This is when $\alpha$ can be dropped and any of the two remaining indices in $I_1^{\mathrm{RR}}$ is non positive. Then the Hypergeometric function reduces to a polynomial in its variable (cf. eqn. (B.14)). The resulting $y$ integral is easily performed and the resulting Hypergeometric and Gamma function(s) can be expanded in $\epsilon$.

Also the integrals for which we are forced to keep $\alpha$, but $a_2 \in -\mathbb{N}_0$ are not too hard to calculate. We expand $[\ldots]^{-a_2}$ in eqn. (7.36), which leads to the factorization of the $u$- and $v$-integrals. By means of eqn. (B.26), each of them will lead to a $\beta$-function. The $y$-integration in $I_2^{\mathrm{RR}}$ will result in a Hypergeometric function via eqn. (B.25).

If all three $a_2$, $a_7$ and $a_1 + a_3$ are positive, the integrals are more difficultbut still feasible, given $\alpha$ can be dropped. Essentially we then use eqn. (7.40) or (7.41), expand the Hypergeometric function containing $y$ to sufficient order in $\epsilon$ and then perform the $y$ integration. This will only lead to a finite result if $a_2 + a_3 + a_5 - 1|_{\epsilon=0} \leq 0$, else $\alpha$ has to be kept.

If $\alpha$ must be kept due to a pole in or in front of the integral, determining the solution is much more involved and will be discussed in the following section.

### General Integral with $\mathbf{a_8, a_9 = 0}$

We will now discuss a general reformulation of any integral with $a_{4,8,9} = 0$. Even though it holds more generally, we will only apply it for those integrals where $\alpha$ has to be kept and $a_2 > 0$, because other integrals can be solved in a simpler way as discussed above. However, the following discussion will also be useful for further classes of integrals.

Continuing from eqn. (7.36), we extract the factor $|\sqrt{u(1-y)} - \sqrt{y(1-u)}|^{-2a_2}$ from $[\ldots]^{-a_2}$ and perform the $v$-integration by using a variant of eqn. (B.25), which then leads to

$$I_1^{\mathrm{RR}}(a_1, a_2, a_3, 0) = \int_0^1 \mathrm{d}u\, u^{-a_3-\epsilon}(1-u)^{-a_1-\epsilon} \left|\sqrt{u(1-y)} - \sqrt{y(1-u)}\right|^{-2a_2}$$
$$\times {}_2F_1\left(a_2, \frac{1}{2} - \epsilon; 1 - 2\epsilon; -\frac{4\sqrt{u(1-u)y(1-y)}}{\left(\sqrt{u(1-y)} - \sqrt{y(1-u)}\right)^2}\right) . \tag{7.44}$$

We split the integral over $u$ in two parts, one from 0 to $y$, the other from $y$ to 1. For both integrals we use eqn. (B.21), which requires $|z| < 1$, to receive a ${}_2F_1$ with a simpler argument. For the first integral we use the identity with $z = -\sqrt{\frac{u(1-y)}{y(1-u)}}$; for the second integral we use $z = -\sqrt{\frac{y(1-u)}{u(1-y)}}$. To receive

$$
\begin{aligned}
I_1^{\mathrm{RR}}(a_1,a_2,a_3,0) = &\int_0^y \mathrm{d}u\, y^{-a_2} u^{-a_3-\epsilon}(1-u)^{-a_1-a_2-\epsilon}{}_2F_1\Big(a_2,a_2+\epsilon;1-\epsilon;\tfrac{u(1-y)}{y(1-u)}\Big) \\
&+ \int_y^1 \mathrm{d}u\,(1-y)^{-a_2} u^{-a_2-a_3-\epsilon}(1-u)^{-a_1-\epsilon}{}_2F_1\Big(a_2,a_2+\epsilon;1-\epsilon;\tfrac{y(1-u)}{u(1-y)}\Big) \ . \qquad (7.45)
\end{aligned}
$$

For each of the integrals we substitute $u$ by the last argument of ${}_2F_1$, which we call $t$ and which ranges from 0 to 1. Then we can combine the two summands again to receive

$$
\begin{aligned}
I_1^{\mathrm{RR}}(a_1,a_2,a_3,0) = y^{1-a_2-a_3-\epsilon}(1-y)^{1-a_1-a_2-\epsilon}\int_0^1 \mathrm{d}t\,{}_2F_1(a_2,a_2+\epsilon;1-\epsilon;t) \qquad (7.46)\\
\times\left[t^{-a_3-\epsilon}\{1-y(1-t)\}^{-2+a_1+a_2+a_3+2\epsilon} + t^{-a_1-\epsilon}\{1-(1-y)(1-t)\}^{-2+a_1+a_2+a_3+2\epsilon}\right] .
\end{aligned}
$$

Recall from eqn. (B.17) the definition of $\gamma$ and that the series expansion of ${}_2F_1$ only converges at $t=1$ for $\gamma>0$. Here $\gamma = 1-2a_2-2\epsilon$, which is $\leq -1$ if $a_2$ is a positive integer and we set $\epsilon$ to zero. However, $a_2>0$ is exactly the case, we consider here. We therefore use eqn. (B.19) to extract the pole at $t=1$ and receive a new ${}_2F_1$ with a $\gamma'$ in the appropriate domain. Combining this result with the remaining $I_2^{\mathrm{RR}}$, we receive the final form

$$
\begin{aligned}
I^{\mathrm{RR}}(a_1,a_2,a_3,0,a_5,a_6,a_7,0,0) = \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}t\, y^{1-a_2-a_3-a_5-\epsilon}(1-y)^{1-a_1-a_2-a_6-\epsilon} \\
\times\left[1-z_-(1-y)\right]^{-a_7}(1-t)^{1-2a_2-2\epsilon}{}_2F_1\big(1-a_2-\epsilon,1-a_2-2\epsilon;1-\epsilon;t\big) \qquad (7.47)\\
\times\left[t^{-a_3-\epsilon}\{1-y(1-t)\}^{-2+a_1+a_2+a_3+2\epsilon} + t^{-a_1-\epsilon}\{1-(1-y)(1-t)\}^{-2+a_1+a_2+a_3+2\epsilon}\right] .
\end{aligned}
$$

It is worth noting that the factor $y^{1-a_2-a_3-a_5-\epsilon}$, in contrast to all other factors, is not regulated by $\epsilon$but only by $\alpha$. This is the potential source of poles in $\alpha$ in the integrals. When $1-a_2-a_3-a_5-\epsilon|_{\alpha=0}<0$ such poles in general appear, as was already noted below eqn. (7.43).

For generic $a_i$ eqn. (7.47) cannot be solved in a closed form. However, in section 7.8 we will discuss how this integral is solved in a series in $\alpha$ and $\epsilon$ for explicit values of $a_i$. First however, we will extend the discussion of this section to integrals with $a_8$ or $a_9 \neq 0$ in the next section.

### 7.7.2 Integrals with $a_8$ or $a_9 \neq 0$

Now let us turn to the cases $a_8$ or $a_9 \neq 0$. As discussed in section 7.6.4, for the integrals of that type it is always possible to choose $a_1,a_3,a_4=0$ and $a_8$ or $a_9=0$. Integrals of this kind never have poles in $\alpha$. The simplest way to see this is to consider the corresponding integral with $a_8,a_9=0$ which by eqns. (7.35, 7.40) and ${}_2F_1(0,a_2;c;y)=1$ is free of poles in $\alpha$, because $\epsilon$ is sufficient to regulate all poles. Since $D_8$ and $D_9$ do not have poles for any $z \in [0,1)$, integrals containing them should not have higher poles in $\alpha$ or $\epsilon$ as the corresponding integrals without them. Hence, the integrals considered in this section will be free of poles in $\alpha$. Furthermore they will have no prefactor $\alpha^{-1}$, because this always is joined by a factor $\delta(1-z)$ which would set $a_{i\geq 7}$ to 0. Because we are only interested in the expansion of $\mathcal{B}$ to $\alpha^0$, we can therefore drop $\alpha$ here.

Instead of using $a_1, a_3 = 0$ we will sometimes use negative values for them, as we can avoid some poles in the integration variables in that way. Also for these integrals $\alpha$ can be dropped, as can be seen in an analog way as just discussed.

For generic $a_5, a_6, a_7$ we have three different classes of integrals: Those with $a_2 \leq 0$, those with $a_2, a_8 > 0$ and those with $a_2, a_9 > 0$.

### Simple Cases

Let us first discuss the simple case, where $a_2, a_4 = 0$. We then use eqn. (7.31) to choose $a_9 = 0$. After the substitution of eqn. (7.37) we arrive at

$$\begin{aligned} I^{\mathrm{RR}}(0,0,0,0,a_5,a_6,a_7,a_8,0) &= \frac{2^{-4\epsilon}}{\pi}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v\, u^{-\epsilon}(1-u)^{-\epsilon} v^{-\epsilon-1/2}(1-v)^{-\epsilon-1/2} \\ &\quad \times [1-z_-(1-u)]^{-a_8}\int_0^1 \mathrm{d}y\, y^{-a_5}(1-y)^{-a_6}[1-z_-(1-y)]^{-a_7} \qquad (7.48) \\ &= \frac{\Gamma(1-a_5)\Gamma(1-a_6)\Gamma^2(1-\epsilon)}{\Gamma(2-a_5-a_6)\Gamma(2-2\epsilon)}\,{}_2F_1(a_7, 1-a_6; 2-a_5-a_6; z_-)\,{}_2F_1(a_8, 1-\epsilon; 2-2\epsilon; z_-)\,, \end{aligned}$$

where in the last step we identified a $\beta$-function for the $v$-integral according to (B.26) as well as a ${}_2F_1$-function for each the $u$- and the $y$-integral according to (B.25). This was possible because no factor mixes two integration variables.

Integrals with $a_2 < 0$ could be determined in a similar way after the expansion of $D_2$. However, we did not encounter integrals of that type in our calculation.

### Integrals with $\mathbf{a_2}, \mathbf{a_8} > \mathbf{0}$

For integrals with $a_2, a_8 > 0$, the calculation equals the one of eqn. (7.47) until we arrive at eqn. (7.45). Using this last equation, and combining it with the suppressed $y$ and $D_8$ part, we receive

$$\begin{aligned} I^{\mathrm{RR}}(a_1,a_2,a_3,0,a_5,a_6,a_7,a_8,0) &= \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}y\, y^{-a_5}(1-y)^{-a_6}[1-z_-(1-y)]^{-a_7} \\ &\times [1-z_-(1-u)]^{-a_8}\Big\{\theta(y-u)y^{-a_2}u^{-a_3-\epsilon}(1-u)^{-a_1-a_2-\epsilon}\,{}_2F_1\Big(a_2, a_2+\epsilon; 1-\epsilon; \tfrac{u(1-y)}{y(1-u)}\Big) \\ &+ \theta(u-y)(1-y)^{-a_2}u^{-a_2-a_3-\epsilon}(1-u)^{-a_1-\epsilon}\,{}_2F_1\Big(a_2, a_2+\epsilon; 1-\epsilon; \tfrac{y(1-u)}{u(1-y)}\Big)\Big\}\,. \qquad (7.49) \end{aligned}$$

If both $a_7$ and $a_8$ are positive, we iteratively partial fraction $D_7$ and $D_8$ using

$$\frac{1}{1-z_-(1-y)}\frac{1}{1-z_-(1-u)} = \frac{1}{u-y}\left[\frac{1-y}{1-z_-(1-y)} - \frac{1-u}{1-z_-(1-u)}\right] \tag{7.50}$$

until in each summand only one of the two denominators $D_7$ or $D_8$ is left. Then for each term we introduce the new variable $t(u,v)$ as the argument of ${}_2F_1$, for the terms without $D_8$ we substitute $t$ for $u$ as we did in (7.45) and then rename $y$ as $u$, for the terms without $D_7$, we substitute $t$ for $y$, instead. As $a_2 > 0$, we use again eqn. (B.19) to obtain a ${}_2F_1$ without a pole at $t = 1$.

The cases appearing in our calculation can then be written as

$$\begin{aligned}I^{\mathrm{RR}}(a_1,a_2,a_3,0,a_5,a_6,0,a_8,0) = &\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}t\, u^{1-a_2-a_3-a_5-\epsilon}\big(1-u\big)^{1-a_1-a_2-a_6-\epsilon} \\ &\times \left[1-z_-(1-u)\right]^{-a_8}\big(1-t\big)^{1-2a_2-2\epsilon}{}_2F_1\big(1-a_2-\epsilon,1-a_2-2\epsilon;1-\epsilon;t\big) \\ &\times \left\{t^{-a_6}\left[1-(1-t)(1-u)\right]^{-2+a_2+a_5+a_6} + t^{-a_5}\left[1-(1-t)u\right]^{-2+a_2+a_5+a_6}\right\}.\end{aligned} \tag{7.51}$$

and

$$\begin{aligned}I^{\mathrm{RR}}(0,a_2,0,0,a_5,a_6,1,1,0) = &\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}t\, u^{-a_2-a_5-\epsilon}(1-u)^{1-a_2-a_6-\epsilon}\frac{1}{1-z_-(1-u)} \\ \times \big(1-t\big)^{-2a_2-2\epsilon}{}_2F_1\big(1-a_2-\epsilon,1-a_2-2\epsilon;1-\epsilon;t\big)&\Big\{-t^{-\epsilon}\left[1-(1-t)u\right]^{-1+a_2+2\epsilon} + t^{-\epsilon}\left[1-(1-t)(1-u)\right]^{-1+a_2+2\epsilon} \\ + t^{-a_6}\left[1-(1-t)(1-u)\right]^{-1+a_2+a_5+a_6} - t^{-a_5}&\left[1-(1-t)u\right]^{-1+a_2+a_5+a_6}\Big\}\end{aligned} \tag{7.52}$$

As keeping negative $a_1$ or $a_3$ in the second case, does not simplify the latter expression, we put them to 0 there. We will continue the discussion of these integrals in the following sections and also address several complications, which might be obvious for the careful reader already at this point.

### Integrals with $\mathbf{a_2, a_9 > 0}$

Expressions for integrals with $a_2, a_9 > 0$ can be obtained as those of the last subsection: We use eqn. (7.47) with eqn. (7.45) and introduce $D_9 = 1 - z_-u$, to obtain the analog of eqn. (7.49) with $\{a_8, [1-z_-(1-u)]\} \to \{a_9, [1-z_-u]\}$. For $a_7 \neq 0$ the two $z_-$-dependent factors $D_7$ and $D_9$ can be partial fraction decomposed as in (7.50) with $1-u \leftrightarrow u$, the resulting expressions is more involved as the introduced extra denominator mixing the two variables, cannot be expressed in terms of the other denominators. Nevertheless one can proceed analogously as for $a_8 > 0$.

For the relevant case with $a_7 = 0$, we simply obtain analogously to eqn. (7.51)

$$
\begin{aligned}
I^{\mathrm{RR}}(a_1, a_2, a_3, 0, a_5, a_6, 0, 0, a_9) = &\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}t \, u^{1-a_2-a_3-a_5-\epsilon} \big(1-u\big)^{1-a_1-a_2-a_6-\epsilon} \\
&\times \big[1 - z_- u\big]^{-a_9} \big(1-t\big)^{1-2a_2-2\epsilon} {}_2F_1\big(1-a_2-\epsilon, 1-a_2-2\epsilon; 1-\epsilon; t\big) \\
&\times \Big\{ t^{-a_6} \big[1 - (1-t)(1-u)\big]^{-2+a_2+a_5+a_6} + t^{-a_5} \big[1 - (1-t)u\big]^{-2+a_2+a_5+a_6} \Big\} \,.
\end{aligned} \tag{7.53}
$$

For $a_7 \neq 0$ we only have to consider the integral

$$
\begin{aligned}
I^{\mathrm{RR}}\big(0,1,0,0,-\epsilon-\beta,\epsilon,1,0,1\big) = &\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}t \, \big(1-t\big)^{-1-2\epsilon} {}_2F_1\big(-\epsilon,-2\epsilon;1-\epsilon;t\big) \big[1-z_- u\big]^{-1} \\
\times \Big\{ &+ u^{1+\beta} \big(1-u\big)^{-2\epsilon} \big[1 - (1-u)(1-t)\big]^{-\beta} \big[u^2 - t(1-u)^2\big]^{-1} t^{-\epsilon} \\
&- u^{1+\beta} \big(1-u\big)^{-2\epsilon} \big[1 - u(1-t)\big]^{-\beta} \big[(1-u)^2 - tu^2\big]^{-1} t^{\epsilon+\beta} \\
&+ u^{1-2\epsilon} \big(1-u\big)^{\beta} \big[1 - (1-u)(1-t)\big]^{2\epsilon} \big[u^2 - t(1-u)^2\big]^{-1} t^{-\epsilon} \\
&- u^{1-2\epsilon} \big(1-u\big)^{\beta} \big[1 - u(1-t)\big]^{2\epsilon} \big[(1-u)^2 - tu^2\big]^{-1} t^{-\epsilon} \Big\} \,.
\end{aligned} \tag{7.54}
$$

Obviously the unregulated mixed denominator between $u$ and $t$ will complicate the calculation. More detail follow in section 7.8.

### 7.7.3 Appearance of Indices

We discussed already in section 7.4.1 and especially in Table 7.1 how the indices are connected to the amplitude topologies depicted in diagrams (a), (b), (c) in Figure 7.1. Let us recall those connections and the implications of the relations of section 7.6.4 with focus on the integrals, discussed in the last subsections which are not simple cases.

All of those more involved integrals have $a_2 > 0$ and their complexity increases from $a_2 = 1$ to $a_2 = 2$. While eqn. (7.25) is relevant for both parameterization the other cases use only parameterization A.

In that parameterization $D_2^{-1}$ is induced by $1/(p-l)^2$. Through the renaming $l \leftrightarrow k-l$, which is used in eqn. (7.31), it can also be created by $1/(p-k+l)^2$. Those two denominators are partial fraction decomposed with each other. Hence, in parameterization C $a_2 = 2$ only appears for diagram (a) or (b) combined with itself. $a_2 = 1$ occurs only if at least one of them is present.

In parameterization C, in contrast, $a_2$ enters via $1/\bar{n} \cdot l$ or after renaming via $1/\bar{n} \cdot (k-l)$. They appear only through an external gluon with the corresponding momentum. As the two denominators are partial fraction decomposed with each other, the maximal value in parameterization C is $a_2 = 1$.

In all combinations of diagrams in principle $a_7 = 2$ could appear as $D_7^{-1}$ enters in both parametrizations through $1/(p-k)^2$ which is present one time in all diagrams (a,b,c). In the actual calculation of integrals with $a_2 > 0$ to $\alpha^0$, we however only have to deal with

$a_7 = 1$ or 0.

In presence of $D_8$ and $D_9$ always parameterization A is chosen. $D_8^{-1}$ is introduced through $1/\bar{n} \cdot (p-l)$ or through $1/\bar{n} \cdot (p-k+l)$ after renaming. This requires diagram (a) or (b) with a gluon at the corresponding propagator. $D_2^{-1}$ is introduced by the same propagator. To find $a_8 = 2$, diagram (a) or (b) has to be combined with itself. Note that if $1/\bar{n} \cdot (p-l)$ and $1/\bar{n} \cdot (p-k+l)$ are present, they are partial fraction decomposed as are $D_2^{-1}$ and $D_4^{-1}$. Then eqn. (7.31) is used to assure $a_4 = 0$ or in case of $a_2, a_4 = 0$ to choose $a_9 = 0$. Therefore we never will find $a_9 = 2$ and $a_9 = 1$ only appears with $a_2 = 1$ through the crossed combination of diagram (a) and (b) with a gluon at both propagators $p-l$ and $p-k+l$.

### 7.7.4 Simpler Basis for Integrals

Before we discuss in section 7.8 in some more detail, how we solved the complicated integrals with $a_2 > 0$ for explicit choices of the indices, let us design an approach to reduce the number and complexity of these integrals.

#### For $\mathbf{a_8, a_9 = 0}$

Let us start with the integrals with $a_8, a_9 = 0$. For the complicated cases for which $a_2 > 0$ and $\alpha$ has to be kept, we will use representation (7.47) for the calculation of such integrals. In the other cases, the simpler approaches of section 7.7.1 are applicable.

We observed in section 7.7.1 that due to the presence of $y^{1-a_2-a_3-a_5-\epsilon}$ these integrals contain poles in $\alpha$ exactly if $a_2 + a_3 + a_5 - 1 > 0$.[2] All other poles are regulated by $\epsilon$. Furthermore, it is obvious from eqn. (7.47) that for given other indices the integral is simplest for $a_7 = 0$ as the related denominator does not appear then. For this reason we obviously want to avoid both. Of course this is not possible for all integrals, however, we can avoid to have both of them at the same time.

As the power of $y$ is the only place where $a_5$ appears, an elegant way to do so is to PF $a_5$ and $a_7$ until either $a_7 = 0$ or $a_5 \leq 1 - a_2 - a_3$.[2] The same trick can be used with $a_6$ and $a_7$ to avoid a pole at $1-y$. We PF them until either $a_7 = 0$ or $a_6 \leq 1 - a_1 - a_2$.[2] There will never be a pole in $\alpha$ at $y = 1$, but we can avoid a pole in $\epsilon$ and simplify the leftover integral with $a_7 > 0$ as much as possible. Moreover, if we would not PF the last two factors here, this would be exactly what we do within the calculation of the corresponding integral. Hence in our approach, we are left with only one of those terms, while the others are separated of. Some of those integrals might even be shared among different terms and it is in this sense that we reduce the workload.

Then we have either integrals with $a_7 = 0$ or integrals without poles in $\alpha$. For the latter ones $\alpha$ can be dropped completely, because our steps from section 7.6.4 guarantee that no $\alpha$ poles appear in front of an integral with $a_7 > 0$. If this leads to a smaller value of $a_2$, we use eqn. (7.30) or (7.32) and then readjust $a_5$ and $a_6$ again. In this way some of the

[2] This is meant up to regulators.

$a_7 > 0$ integrals become one of the simple cases discussed early in section 7.7.1. The $a_7 > 0$ integrals which do not turn to those simple cases, still have the nice property that they have no pole in the whole range of the $y$ integral. This includes the mixed denominator, because with our PF choice a pole there implies $a_1 \leq 0$ which however would have been mapped to a simple $a_2 < 0$ integral by eqn. (7.30), as explained above.

For the integrals with $a_7 = 0$ we continue to PF the indices $a_5$ and $a_6$ until either $a_5 \leq 1 - a_2 - a_3$ or $a_6 \leq 1 - a_1 - a_2$.[2] If we have the choice, we prefer to have the first condition fulfilled. Thus, at least one of them is small enough such that the $y$-integral has at most a pole at 0 or 1.

### For $a_8$ or $a_9 \neq 0$

Let us now consider a generic integral with $a_2 > 0$ and $a_8$ or $a_9 > 0$. As discussed in sections 7.6.4 and 7.7.2 this contains all non trivial cases and they will never be multiplied by $\alpha^{-1}$.

Let us first consider a generic integral with $\mathbf{a_2, a_8 > 0}$, but with $a_7 = 0$. Due to PF of section 7.6.4 we then chose $a_1, a_3, a_4, a_9 = 0$ and eqn. (7.51) is the relevant parametrization. We argued in the beginning of section 7.7.2 that $\alpha$ can be dropped for these integrals. Nevertheless, the power $1 - a_2 - a_3 - a_5 - \epsilon$ of $u$ contains no regulator and can easily be negative. We therefore assumed a regulator $-\beta$ on $a_5$ for intermediate steps. In the final result, there will no pole be left in this regulator and it can be dropped.

These integrals are very similar to those we discussed in the last paragraph. One difference is the naming $u \leftrightarrow y$ and the labeling $(a_8, a_1, a_3, a_5, a_6) \leftrightarrow (a_7, a_6, a_5, a_3, a_1)$. The other difference is the exact occurrences of the regulators. For example the mixed denominator will only contain the intermediate regulator $\beta$, which often allows us to achieve a factorization of the $u$ and $t$ integrals.

Anyway, as they are very similar, we can use a similar approach to map them to simpler integrals. This is we PF $a_8$ with $a_3$ and $a_1$, pushing the later ones to negative values, until either $a_8 = 0$ or $a_3 = 1 - a_2 - a_5|_{\epsilon=0}$ and $a_1 = 1 - a_2 - a_6|_{\epsilon=0}$. The remaining integrals with $a_8 > 0$ have no poles in $u = 0, 1$. If the mixed denominator between $u$ and $t$ appears at non-negative power, i.e. $-2 + a_2 + a_5 + a_6 \geq 0$, we can drop the intermediate regulator $\beta$ and expand this denominator such that the $u$ and $t$ integral factorize. We also found two cases with negative power of these denominators. Also for them it turned out that the regulator $\beta$ can be dropped.

The integrals with $a_8 = 0$ generated by this procedure are simple. For reasons discussed above the regulator $\alpha$ is not needed for them. Then eqn. (7.30) can be used followed by partial fractioning of $a_2$ and $a_4$ to obtain integrals of the form (7.40) with $a_2 \leq 0$. As the new $a_2$ is a non-positive integer, the Hypergeometric function is just a polynomial in $y$ (cf. eqn. (B.16)), which can be expanded, such that each remaining $y$ integral just leads to a $\beta$-function.

In exactly the same way the integrals with $\mathbf{a_2, a_9 > 0}$ and $a_4, a_7, a_8 = 0$ can be treated. The only difference are the PF relations which are essentially exchanged between $a_1$ and $a_3$ w.r.t. the $a_8 > 0$ case.

Then the only remaining tough integrals with $\mathbf{a_8}$ **or** $\mathbf{a_9} > \mathbf{0}$ are those where in addition $\mathbf{a_2}, \mathbf{a_7} > \mathbf{0}$. There appear only very few ones, but their calculation is quite involved. Let us have an exemplary view on eqn. (7.52). On the first view this looks a lot like being analogue to the cases considered above, however, there are some unpleasant differences, hampering us from playing the same trick again. Again the power $-a_2 - a_3 - a_5 - \epsilon$ is not regulated by epsilon. Having dropped $\alpha$ earlier, we again understand $a_5$ to contain a regulator $\beta$. What prevents us from simplifying the expression to one with either $a_{i\geq 7} = 0$ or without poles in $u$, is the fact that the mixed denominators now come in pairs with different powers. We therefore stick to the indices obtained form the tactics of section 7.6.4 for those integrals.

To sum up the results of this section once more: Apart from the last type of integrals, we discussed a tactic which guarantees that

- we only find $\alpha$ poles in integrals with $a_{i\geq 7} = 0$,
- integrals with $a_7 > 0$ (but $a_8, a_9 = 0$) have no pole in $y \in [0, 1]$,
- integrals with $a_8$ or $a_9 > 0$ (but $a_7 = 0$) have no pole in $u = 0, 1$, besides $\epsilon$ no regulator is needed for them.

## 7.8 Calculation of Explicit Integrals

In this section, we discuss the calculation and especially the expansion in $\alpha$ and $\epsilon$ of the integrals $I^{\mathrm{RR}}$ for given explicit indices. As has been discussed in the second part of section 7.5, they have to be expanded to the order $(\alpha/\epsilon)^1$, $\epsilon^2$ in presence of $\delta(z_-)$ and to $(\alpha/\epsilon)^0$, $\epsilon^1$ in other cases. For integrals appearing only for $\mathcal{B}'$ the expansion to $(\alpha/\epsilon)^0$ and $\epsilon^0$ is sufficient. We will first comment on the expansion of the simple integrals. After that we will discuss the calculation of the more involved integrals.

### 7.8.1 Simple Cases

As simple cases we refer to all integrals, for which we either gave a solution in closed form or described the algorithm how to obtain it for given indices. With the maps between several integrals used as described in sections 7.6.4 and 7.7.4, this includes all integrals discussed in the first subsection of section 7.7.1 and integrals with $a_8 > 0$ but $a_2 = 0$.

In most of these cases $\alpha$ has been dropped to arrive at the closed form. Then the closed form already corresponds to an expansion in $\alpha$ and the expansion is in $\epsilon$ only. For the other cases we expand in both regulators, with the expansion in $\alpha$ taken first. The required order is stated above. The expansion is taken starting from the corresponding closed from with explicit indices. $\Gamma$-functions can straight forwardly be expanded. Some of them will contain poles in $\epsilon$. For Hypergeometric functions, we first use eqns. (B.15) or (B.16) if applicable. If needed, we also use eqn. (B.19). Then the expansion can be taken by Mathematica with the help of the package HypExp. For the final result we will map occurrences of generalized harmonic polylogarithms to the form $H_{\vec{n}_i}(z)$ and expand products of several $H$ functions by means of eqn. (B.10).

### 7.8.2 Further Cases

In section 7.7.4, we described, how to choose the indices of the integrals with $a_8$ or $a_9 > 0$, but $a_7 = 0$, in an improved way, such that we do not need to keep the intermediate regulator *beta* for them. Here, we discuss their calculation.

If $-2+a_2+a_5+a_6 \geq 0$ we can expand the mixed denominator between $t$ and $u$, such that the two integrals factorize. For each term the $u$ integral is then straight forwardly taken to give a Gauss Hypergeometric function via eqn. (B.25). As our algorithm guaranteed the absence of poles at $u = 0, 1$, a range in $\epsilon$ located around 0 exists, where the constraints are fulfilled.

Similarly for each term the $t$ integral can be taken in terms of a generalized Hypergeometric function via eqn. (B.22). However, in many cases, with $a2 = 2$ for example, no consistent range in epsilon exists, where the powers of both $(1-t)$ and $t$ are larger than $-1$. Then we first partial fraction the denominator $(1-t)$ with $t$ and iterate this with integration by parts for $(1-t)$ until for each term $\epsilon$ can be chosen in a range located around 0. Then we can securely use eqn. (B.22) to perform the $t$-integrals in terms of ${}_3F_2(\ldots,1)$.

The obtained results are in a closed form and we can expand them in the same way as discussed in the last subsection. Expansions of ${}_3F_2(\ldots,1)$ can be taken by Mathematica.

If $-2+a_2+a_5+a_6 < 0$ the mixed denominator between $t$ and $u$ is present. The only two cases which appeared were up to regulators all indices vanish but $a_2 = 1$ and $a_8 = 1$ or $a_9 = 1$. For them the Hypergeometric function present is ${}_2F_1({\scriptstyle -\epsilon,-2\epsilon;1-\epsilon;t}) = 1 + \mathcal{O}(\epsilon^2)$. We partial fraction the mixed denominator with $(1-t)^{-n}$ until only one of them is left. The terms without the mixed denominator are of the same for as for the integrals discussed before with a proper range in $\epsilon$ and can therefore be solved straightforwardly. The terms with the mixed denominator have no pole in any of $t, u = 0, 1$ which implies that no powers of $\epsilon^{-1}$ can be generated by them. As the expansion of the integral is needed to $\epsilon^1$ only, we can then expand the Hypergeometric function to order $\epsilon^1$ which effectively sets it to 1. Then the $t$ integral including the mixed denominator can be taken via eqn. (B.25). Its result and the remaining factor expanded to $\epsilon^1$, such that the $u$-integral can be performed in Mathematica.

### 7.8.3 Complicated Cases

We now will discuss the calculation of the complicated integrals where we use one of the parametrizations (7.25), (7.52) or (7.54). The complicated integrals cannot be solved in a closed form in terms of standard functions, but we will derive their expansion in terms of the regulators to sufficient order.

Let us start with the integrals with $\mathbf{a_8}, \mathbf{a_9} = \mathbf{0}$. Due to the presence of two regulators, the Hypergeometric function, the mixed denominator between $t$ and $y$ and the denominator containing $z$, the integral is quite difficult to solve. Due to the work of last section the $y$-integral will have only a pole in 0 or 1 or contain $D_7$. Moreover, for the cases relevant for us poles in $y = 0, 1$ are at most of first order. We partial fraction the denominators containing $t$ until for each term there is at most a pole at 0 or 1 in the $t$-integral. If there are poles of higher order in $t$ or $y$, we continue partial fraction the denominators until $\epsilon$ of each individual term can be chosen such that at most single poles are present. Then we use integration by parts iterated with PF to arrive at terms, where $\epsilon$ can be chosen infinitesimal. This can increase the number of terms drastically.

For each term we split the factor carrying a pole via eqn. (B.32) in terms of a $\delta$- and a plus-distribution. The $\delta$-terms can be solved analytically. The remaining terms need to be expanded in the regulators before integration. Polylogarithms of the integration variable are mapped to arguments such that they can compensate an eventual pole of the integral. The relevant relations can be found in [39] for example.

Then the integrals for each regular combination of terms can be solved, e.g. by Mathematica. After the first integration has been performed, in some cases the steps of expanding and proper choice of arguments of Polylogarithms have to be repeated before the last integral is performed.

For integrals with $a_7 = 0$ and no pole at $y = 0$ as well as at most a pole of first order at 1, we usually perform the $y$ integral analytically by means of eqn. (B.25). Then we continue similarly as discussed above. For some cases this turns out to be inconvenient, then we use the standard method described above.

If the $y$ integration is not been taken in terms of a Hypergeometric function, the mixed denominator in eqn. (7.47) could cause trouble, when its power becomes non positive. The only integral relevant for us with such trouble is $I^{\mathrm{RR}}(\alpha, 1, 1+\alpha, 0, -1-\epsilon-\alpha(1+x), 1+\epsilon, 0, 0, 0)$. The power of the mixed denominator in this case is $x = 2(\alpha+\epsilon)$, which seems unproblematic. However, we find among others the term

$$t^{-1-\alpha-\epsilon}(1-y)^{-1-\alpha-2\epsilon}[1-(1-t)y]^{2\alpha+2\epsilon} f_{\alpha,\epsilon}(t,y) \tag{7.55}$$

which contains as many of the other integrals a pole at $t = 0$ as well as at $y = 1$, which on first sight seem to be nicely regulated by $|\alpha| < |\epsilon|$ and $\epsilon < 0$. However, consider the last factor: At $y = 1$ it reduces to $t^{2\alpha+2\epsilon}$ and at $t = 0$ to $(1-y)^{2\alpha+2\epsilon}$. But this inverts the allowed range of regulators on the other pole. We therefore in a first step subtract this double pole by adding 0 in the form $t^{-1-\alpha-\epsilon}(1-y)^{-1-\alpha-2\epsilon}(1-(1-t)y)^{2\alpha+2\epsilon}(f_{\alpha,\epsilon}(0,0) - f_{\alpha,\epsilon}(0,0))$. The second summand can be combined with the problematic term (7.55) to cure the problematic double pole. The first term can be calculated analytically with the help of (B.22). For the curious reader, after PF we have above $f_{\alpha,\epsilon}(t,y) = (1-t)^{-2\epsilon} y^{\alpha n} {}_2F_1(-2\epsilon, -\epsilon; 1-\epsilon; t)$, where $n = 1$ for the collinear and $n = -1$ for the anti-collinear case.

The calculation of the integrals with $\mathbf{a_2, a_7, a_8 > 0}$ is performed very similar to the integrals with $a_8, a_9 = 0$. Despite the higher number of terms and the changed variable name $u \leftrightarrow y$, there are two main differences. We drop $\alpha$ but introduce for intermediate steps the new regulator $-\beta$ on $a_5$ if $a_2 + a_5 \geq 1$. Of course, this is only sensible if the full integral has no poles in that regulator. The other difference is that in addition to the denominators containing $t$, we also usually have to partial fraction the denominators containing $u$ until for each term there is at most a pole at 0 or 1 left and terms with poles in $u$ do not contain $D_8$. In one case also for $u$ iterated additional partial fractioning and integration by parts is needed to be allowed to choose $\epsilon$ infinitesimally. The remaining steps are analogue to what we explained for the other case.

The integral with $\mathbf{a_2}, \mathbf{a_7}, \mathbf{a_9} > \mathbf{0}$ gains yet another complication with respect to those integrals, which are the additional mixed denominators between $u$ and $t$. We decompose this denominators in terms of $y$ and $y_\pm = \frac{1\pm\sqrt{t}}{1-t}$ and then partial fraction those denominators with each other and the remaining denominators until in each term only one denominator with negative power will be left for $t$ and $y$. Most of those terms can than be calculated analogously as for the other integrals. For some terms special care is needed as they have a pole within the range of integration. For those we restrict to the principal parts of the integrals. Then we can proceed analogously as for the other integrals.

The final results for all non-trivial integrals we will provide in section D for the cases which appeared. If it contains a $z$ dependence, we express it in terms of powers of $z$ or $z_-$ and harmonic polylogarithms. Products of several harmonic polylogarithms are expanded by means of eqn. (B.10).

## 7.9 Bare Results

In the last sections, we have seen how we can solve all appearing integrals of eqn. (7.1). We can therefore extract the RR result. As discussed in section 4.8, we replace occurrences of the bare coupling constant by the renormalized one. For the RR-contribution to the $\alpha_s^2$ term this leads to the inclusion of the squared $\overline{\text{MS}}$-factor $[\mu^2 e^{\gamma_e}/(4\pi)]^{2\epsilon}$. The total expression is then expanded up to $\alpha^0$ and $\epsilon^0$. Occurrences of $x_T^2$ are expressed via eqn. (4.33) in terms of $L_\perp$ and appearances of the analytic regulator scale $v$ are expressed via eqn. (4.35) by $L_c$ in the collinear region and via eqn. (4.34) by $L_a$ in the anti-collinear region.

We can then extract the results for the real-real contribution to the (nT)PDFs. For the PDFs, we again find the simple result

$$\phi_{i/j}^{(RR)}(z, \mu) = 0\,. \tag{7.56}$$

To present the results for the nTPDFs in a compact form, we extract the mass dependent logarithms $L_\perp$, $L_a$ and $L_c$. We then write the individual results as

$$\mathcal{B}_{i/j}^{(RR)}(z, x_T^2, \mu, v) = \exp[2\alpha L_c + 2\epsilon L_\perp]\, f_{i/j}^{RR}(z, 1) + \mathcal{O}(\alpha, \epsilon) \tag{7.57}$$

and

$$\bar{\mathcal{B}}_{i/j}^{(RR)}(z, x_T^2, \mu, v) = \exp[2\alpha L_a + 2\epsilon L_\perp]\, f_{i/j}^{RR}(z, -1) + \mathcal{O}(\alpha, \epsilon)\,. \tag{7.58}$$

The functions $f_{i/j}^{RR}$ contain the results for the evolution of a parton $j$ to a parton $i$ with two unresolved emissions. Potential flavors of the partons $i$ and $j$ are fixed. If $i = j$, the function $f$ receives contribution from two separate channels, the unresolved emission of two gluons or alternatively a quark-anti-quark-pair of any flavor. In all other cases, only a single channel contributes. In addition to the two specified parameters, they also depend on $\alpha$ and $\epsilon$. The results can be expressed in terms of harmonic polylogarithms $H_{\vec{a}_n} \equiv H_{\vec{a}_n}(z)$ up to weight 3, $\zeta$-values up to weight 4 and functions $\tilde{p}_{ij}$ related to the lowest order DGLAP splitting kernels by removing an overall factor and the $\delta$-function. These functions are discussed in sections B.2 and C.3, respectively.

We start the list of the result functions with the gluon-gluon splitting. It results into

$$
\begin{aligned}
f^{(RR)}_{g/g}(z,s) = C_a^2\Big\{&\delta(1-z)\Big[\frac{8}{\alpha^2\epsilon^2} + \frac{8}{\alpha^2}\zeta_2 - \frac{8+10s}{\alpha\epsilon^3} - \frac{11s}{3\alpha\epsilon^2} + \frac{1}{\alpha\epsilon}\Big(-\frac{67s}{9} + 4s\zeta_2\Big)\\
&+ \frac{1}{\alpha}\Big(-\frac{11s}{3}\zeta_2 - \frac{404s}{27} + \frac{2(4+23s)}{3}\zeta_3\Big) + \frac{13+14s}{\epsilon^4} + \frac{11(1+s)}{6\epsilon^3} + \frac{1}{\epsilon^2}\Big(-\frac{67(-1+s)}{18} - (16+3s)\zeta_2\Big)\\
&+ \frac{1}{\epsilon}\Big(-\frac{-202+615s}{27} + \frac{-49+31s}{3}\zeta_3 + \frac{11(-3+s)}{6}\zeta_2\Big) + \Big(\frac{-129+230s}{4}\zeta_4 - \frac{2(-607+3071s)}{81}\\
&+ \frac{11(-11+7s)}{9}\zeta_3 - \frac{67(3+s)}{18}\zeta_2\Big)\Big] + \tilde{p}_{gg}(z)\Big[\frac{16s}{\alpha\epsilon^2} + \frac{16s}{\alpha}\zeta_2 - \frac{20+16s}{\epsilon^3} - \frac{22}{3\epsilon^2} + \frac{1}{\epsilon}\Big(-\frac{134}{9} - 4H_{0,0}\\
&+ 8\zeta_2\Big) + \Big(\frac{4(47+4s)}{3}\zeta_3 - \frac{22}{3}\zeta_2 - \frac{808}{27} - 4H_{0,0,0} + 8H_{0,1,0} - 8H_1\zeta_2 - 8H_{1,0,0} - 8H_{1,1,0}\Big)\Big]\\
&+ \tilde{p}_{gg}(-z)\Big[\frac{1}{\epsilon}\Big(8H_{-1,0} - 4H_{0,0} + 4\zeta_2\Big) + \Big(-8H_{-1}\zeta_2 - 16H_{-1,-1,0} + 8H_{-1,0,0} + 16H_{0,-1,0} - 4H_{0,0,0}\\
&- 8H_{0,1,0} + 4\zeta_3\Big)\Big] + \Big[\frac{1}{\epsilon^2}\Big(\frac{-32+64z-32z^2+32z^3}{z}H_1 - \frac{8(1-z)(11+2z+11z^2)}{3z} - 32\Big(\frac{H_1}{1-z}\Big)_+\\
&- 16(1+z)H_0\Big) + \frac{1}{\epsilon}\Big(\frac{134-243z+237z^2-134z^3}{9z} + \frac{2(25-11z+44z^2)}{3}H_0 - 16(1+z)H_{0,0}\Big)\\
&+ \Big(\frac{-32+64z-32z^2+32z^3}{z}H_1\zeta_2 - \frac{701+149z+536z^2}{9}H_0 + \frac{2(-398+354z-383z^2+422z^3)}{9z}\\
&+ \frac{8(1-z)(11-z+11z^2)}{3z}H_{1,0} + \frac{2(25-11z+44z^2)}{3}H_{0,0} - 32\Big(\frac{H_1}{1-z}\Big)_+\zeta_2 - 16(1+z)H_0\zeta_2\\
&- 16(1+z)H_{0,0,0} - 8(1-z)\zeta_2\Big)\Big]\Big\}\\
+ C_aT_fN_f\Big\{&\delta(1-z)\Big[\frac{4s}{3\alpha\epsilon^2} + \frac{20s}{9\alpha\epsilon} + \frac{1}{\alpha}\Big(\frac{4s}{3}\zeta_2 + \frac{112s}{27}\Big) - \frac{2(1+s)}{3\epsilon^3} + \frac{2(-5+8s)}{9\epsilon^2}\\
&+ \frac{1}{\epsilon}\Big(\frac{2(3-s)}{3}\zeta_2 + \frac{28(-2+9s)}{27}\Big) + \Big(\frac{4(-82+587s)}{81} - \frac{4(-11+7s)}{9}\zeta_3 + \frac{2(15+8s)}{9}\zeta_2\Big)\Big]\\
&+ \tilde{p}_{gg}(z)\Big[\frac{8}{3\epsilon^2} + \frac{40}{9\epsilon} + \Big(\frac{8}{3}\zeta_2 + \frac{224}{27}\Big)\Big] + \Big[\frac{1}{\epsilon}\Big(-\frac{4(-13+9z-12z^2+13z^3)}{9z} + \frac{8(1+z)}{3}H_0\Big)\\
&+ \Big(-\frac{4(-83+72z-96z^2+83z^3)}{27z} + \frac{8(1+z)}{3}H_{0,0} + \frac{4(13+10z)}{9}H_0\Big)\Big]\Big\}\\
+ C_fT_fN_f\Big\{&\Big[\frac{1}{\epsilon^2}\Big(\frac{4(1-z)(4+7z+4z^2)}{3z} + 8(1+z)H_0\Big) + \frac{1}{\epsilon}\Big(-\frac{8(1-z)(1-8z+z^2)}{3z} + 4(3+z)H_0\\
&+ 8(1+z)H_{0,0}\Big) + \Big(-\frac{8(1-z)(1-23z+z^2)}{3z} + \frac{4(1-z)(4+7z+4z^2)}{3z}\zeta_2 + 24(1+z)H_0
\end{aligned}
$$

$$+ 8(1+z)H_0\zeta_2 + 4(3+z)H_{0,0} + 8(1+z)H_{0,0,0}\Big)\Big]\Big\}. \tag{7.59}$$

The function relevant for the splitting of a quark to a gluon is

$$\begin{aligned}
f_{g/q}^{(RR)}(z,s) &= C_f C_a \Big\{ \tilde{p}_{gq}(z)\Big[\frac{8s}{\alpha\epsilon^2} + \frac{8s}{\alpha}\zeta_2 - \frac{6+8s}{\epsilon^3} + \frac{1}{\epsilon^2}\Big(-\frac{31}{3} - 16H_1\Big) + \frac{1}{\epsilon}\Big(\frac{143}{9} + 6\zeta_2\Big) \\
&+ \Big(\frac{8(12+s)}{3}\zeta_3 + \frac{13}{3}\zeta_2 + \frac{44}{3}H_{1,0} - \frac{664}{27} + 8H_{0,1,0} - 20H_1\zeta_2 - 4H_{1,0,0} - 4H_{1,1,0}\Big)\Big] + \tilde{p}_{gq}(-z)\Big[\frac{4}{\epsilon}H_{-1,0} \\
&+ \Big(-4H_{-1}\zeta_2 - 8H_{-1,-1,0} + 4H_{-1,0,0} + 8H_{0,-1,0}\Big)\Big] + \Big[-\frac{8sz}{\alpha\epsilon} + \frac{1}{\epsilon^2}\Big(\frac{-14+55z+8z^2+24sz}{3} \\
&- 4(4+z)H_0\Big) + \frac{1}{\epsilon}\Big(-\frac{56-z+88z^2}{9} + \frac{2(36+9z+8z^2)}{3}H_0 - 4(2+z)H_{0,0} + 16zH_1 - 8\zeta_2\Big) \\
&+ \Big(-\frac{22+z+8z^2}{3}\zeta_2 + \frac{4(1+13z+152z^2)}{27} - \frac{4(2+5z+4z^2)}{3}H_{1,0} + \frac{2(36+9z+8z^2)}{3}H_{0,0} \\
&- \frac{2(249-6z+88z^2)}{9}H_0 + 4H_{-1,0}z - 4(4+z)H_0\zeta_2 - 4(2+z)H_{0,0,0} + 16H_{0,1,0} - 8\zeta_3\Big)\Big]\Big\} \\
&+ C_f^2\Big\{\tilde{p}_{gq}(z)\Big[-\frac{4}{\epsilon^3} - \frac{6}{\epsilon^2} + \frac{1}{\epsilon}\Big(-16 - 4\zeta_2\Big) + \Big(-32 - \frac{8}{3}\zeta_3 - 6\zeta_2\Big)\Big] + \Big[\frac{1}{\epsilon^2}\Big(4+3z+2(2-z)H_0\Big) \\
&+ \frac{1}{\epsilon}\Big(1+11z-(4+3z)H_0 + 2(2-z)H_{0,0}\Big) + \Big(10+15z-5(3-z)H_0 + 2(2-z)H_0\zeta_2 \\
&- (4+3z)H_{0,0} + 2(2-z)H_{0,0,0} + (4+3z)\zeta_2\Big)\Big]\Big\}.
\end{aligned} \tag{7.60}$$

For a gluon evolving to a quark, we instead obtain

$$\begin{aligned}
f_{q/g}^{(RR)}(z,s) &= C_a T_f\Big\{\tilde{p}_{qg}(z)\Big[-\frac{2}{\epsilon^3} - \frac{37}{3\epsilon^2} + \frac{1}{\epsilon}\Big(4H_{1,0} - \frac{44}{3}H_0 - 2\zeta_2 - \frac{4}{9}\Big) + \Big(\frac{56}{3}H_{1,0} - 4H_{1,1,0} - \frac{44}{3}H_{0,0} \\
&- 4H_1\zeta_2 + \frac{136}{9}H_0 - \frac{4}{3}\zeta_3 + \frac{7}{3}\zeta_2 - \frac{964}{27}\Big)\Big] + \tilde{p}_{qg}(-z)\Big[\frac{4}{\epsilon}H_{-1,0} + \Big(4H_{-1,0,0} - 8H_{-1,-1,0} + 8H_{0,-1,0} \\
&+ 4H_{-1,0} - 4H_{-1}\zeta_2\Big)\Big] + \Big[\frac{1}{\epsilon^2}\Big(4(1+4z)H_0 + \frac{8+43z-14z^2}{3z}\Big) + \frac{1}{\epsilon}\Big(4(1+2z)H_{0,0} + \frac{2(19-20z)}{3}H_0 \\
&+ 8z\zeta_2 - \frac{4(16-28z+11z^2)}{9z}\Big) + \Big(4(1+2z)H_{0,0,0} - 16zH_{0,1,0} - \frac{8(2+4z+z^2)}{3z}H_{1,0} + \frac{2(19-32z)}{3}H_{0,0} \\
&- 4H_{-1,0} + 4(1+4z)H_0\zeta_2 + \frac{4(-13+110z)}{9}H_0 + 8z\zeta_3 + \frac{-8+23z+2z^2}{3z}\zeta_2 + \frac{4(68+124z+49z^2)}{27z}\Big)\Big]\Big\} \\
&+ C_f T_f\Big\{\tilde{p}_{qg}(z)\Big[\frac{8s}{\alpha\epsilon^2} + \frac{8s}{\alpha\epsilon} + \frac{8s}{\alpha}(1+\zeta_2) - \frac{8(1+s)}{\epsilon^3} + \frac{1}{\epsilon^2}\Big(-16H_1 - 14 - 8s\Big) + \frac{1}{\epsilon}\Big(-4H_{1,0} - 4H_{0,0} \\
&- 16H_1 + 8\zeta_2 - 26 - 8s\Big) + \Big(-4H_{0,0,0} - 4H_{1,0,0} - 4H_{1,0} - 4H_{0,0} - 16H_1\zeta_2 - 16H_1 + \frac{8(13+s)}{3}\zeta_3 \\
&+ 2\zeta_2 - 50 - 8s\Big)\Big] + \Big[-\frac{8s}{\alpha\epsilon} - \frac{8s}{\alpha} + \frac{1}{\epsilon^2}\Big(2(1-2z)H_0 + 7 + 4z + 8s\Big) \\
&+ \frac{1}{\epsilon}\Big(2(1-2z)H_{0,0} + 16H_1 + (1+4z)H_0 + 20 - z + 8s\Big) + \Big(2(1-2z)H_{0,0,0} + 4H_{1,0} \\
&+ (5+4z)H_{0,0} + 2(1-2z)H_0\zeta_2 + 16H_1 + (8+7z)H_0 + (-9+4z)\zeta_2 + 39 - 3z + 8s\Big)\Big]\Big\}.
\end{aligned} \tag{7.61}$$

In the case of quark to quark splitting, the two subchannels combine into

$$
\begin{aligned}
f_{q/q}^{(RR)}(z,s) = C_f C_a \Big\{ & \delta(1-z)\Big[ -\frac{2s}{\alpha\epsilon^3} - \frac{11s}{3\alpha\epsilon^2} + \frac{s}{\alpha\epsilon}\Big(4\zeta_2 - \frac{67}{9}\Big) + \frac{s}{\alpha}\Big(\frac{38}{3}\zeta_3 - \frac{11}{3}\zeta_2 - \frac{404}{27}\Big) \\
& + \frac{1+2s}{\epsilon^4} + \frac{11(1+s)}{6\epsilon^3} + \frac{1}{\epsilon^2}\Big( -(4-s)\zeta_2 + \frac{67(1-s)}{18}\Big) + \frac{1}{\epsilon}\Big(\frac{-49+31s}{3}\zeta_3 - \frac{11(3-s)}{6}\zeta_2 \\
& + \frac{202-615s}{27}\Big) + \Big(\frac{-37+242s}{4}\zeta_4 - \frac{11(11-7s)}{9}\zeta_3 - \frac{67(3+s)}{18}\zeta_2 + \frac{2(607-3071s)}{81}\Big)\Big] \\
& + \tilde{p}_{qq}(z)\Big[ -\frac{2}{\epsilon^3} - \frac{11}{3\epsilon^2}\frac{1}{\epsilon}\Big( -4H_{1,0} - 2H_{0,0} - \frac{11}{3}H_0 - \frac{67}{9}\Big) + \Big( -2H_{0,0,0} - 4H_{0,1,0} + 4H_{1,1,0} \\
& - \frac{11}{3}H_{0,0} + 4H_1\zeta_2 - \frac{76}{9}H_0 + \frac{2}{3}\zeta_3 - \frac{11}{3}\zeta_2 - \frac{404}{27}\Big)\Big] + \Big[\frac{2-2z}{\epsilon^2} + \frac{1}{\epsilon}\Big( -6 + 4z - 2(1+z)H_0\Big) \\
& + \Big( -4zH_{0,0} + 2(1+5z)H_0 - 4(1-z)\zeta_2 + \frac{2(19-25z)}{3}\Big)\Big]\Big\} \\
+ C_f^2 \Big\{ & \delta(1-z)\Big[\frac{8}{\alpha^2\epsilon^2} + \frac{8}{\alpha^2}\zeta_2 - \frac{8(1+s)}{\alpha\epsilon^3} + \frac{8(1+s)}{3\alpha}\zeta_3 + \frac{12(1+s)}{\epsilon^4} - \frac{4(3+s)}{\epsilon^2}\zeta_2 - (23+3s)\zeta_4\Big] \\
& + \tilde{p}_{qq}(z)\Big[\frac{8s}{\alpha\epsilon^2} + \frac{8s}{\alpha}\zeta_2 - \frac{8(1+s)}{\epsilon^3} + \frac{1}{\epsilon}\Big(4H_{1,0} + 3H_0 + 4\zeta_2\Big) + \Big( -8H_{1,1,0} - 4H_{1,0,0} + 8H_{0,1,0} + 3H_{0,0} \\
& - 8H_1\zeta_2 + 8H_0\frac{4(23+2s)}{3}\zeta_3\Big)\Big] + \Big[ -\frac{8(1-z)s}{\alpha\epsilon} + \frac{1}{\epsilon^2}\Big( -32\Big(\frac{H_1}{1-z}\Big)_+ + 16(1+z)H_1 + 2(1+z)H_0 \\
& + 4(1+2s)(1-z)\Big) + \frac{1}{\epsilon}\Big(2(1+z)H_{0,0} + 16(1-z)H_1 + (3+7z)H_0 + 18 - 16z\Big) + \Big(2(1+z)H_{0,0,0} \\
& + 4(1-z)H_{1,0} + (3+7z)H_{0,0} - 32\Big(\frac{H_1}{1-z}\Big)_+\zeta_2 + 2(1+z)(8H_1+H_0)\zeta_2 + (2-24z)H_0 - 20 + 24z\Big)\Big]\Big\} \\
+ C_f T_f N_f \Big\{ & \delta(1-z)\Big[\frac{4s}{3\alpha\epsilon^2} + \frac{20s}{9\alpha\epsilon} + \frac{1}{\alpha}\Big(\frac{4s}{3}\zeta_2 + \frac{112s}{27}\Big) - \frac{2(1+s)}{3\epsilon^3} + \frac{2(-5+8s)}{9\epsilon^2} \\
& + \frac{1}{\epsilon}\Big(\frac{2(3-s)}{3}\zeta_2 + \frac{28(-2+9s)}{27}\Big) + \Big(\frac{4(11-7s)}{9}\zeta_3 + \frac{2(15+8s)}{9}\zeta_2 + \frac{4(-82+587s)}{81}\Big)\Big] \\
& + \tilde{p}_{qq}(z)\Big[\frac{4}{3\epsilon^2} + \frac{1}{\epsilon}\Big(\frac{4}{3}H_0 + \frac{20}{9}\Big) + \Big(\frac{4}{3}H_{0,0} + \frac{20}{9}H_0 + \frac{4}{3}\zeta_2 + \frac{112}{27}\Big)\Big] - \frac{4(1-z)}{3}\Big\}. \qquad (7.62)
\end{aligned}
$$

In contrast to the single emission, discussed in the last two chapters, for the double emission also the flavor off-diagonal channel and the quark-to-anti-quark channel open. They result into

$$
\begin{aligned}
f_{q/q'}^{(RR)}(z,s) = C_f T_f \Big\{ \Big[ & \frac{1}{\epsilon^2}\Big(4(1+z)H_0 + \frac{2(1-z)(4+7z+4z^2)}{3z}\Big) \\
& + \frac{1}{\epsilon}\Big(4(1+z)H_{0,0} - \frac{2(3+3z+8z^2)}{3}H_0 - \frac{4(1-z)(16-11z+16z^2)}{9z}\Big) \\
& + \Big(4(1+z)H_{0,0,0} - \frac{8(1-z)(2-z+2z^2)}{3z}H_{1,0} - \frac{2(3+3z+8z^2)}{3}H_{0,0} + 4(1+z)H_0\zeta_2 \\
& + \frac{4(21-12z+32z^2)}{9}H_0 - \frac{2(1-z)(4-11z+4z^2)}{3z}\zeta_2 + \frac{2(1-z)(136-71z+208z^2)}{27z}\Big)\Big]\Big\}, \qquad (7.63)
\end{aligned}
$$

and

$$
\begin{aligned}
f_{q/\bar{q}}^{(RR)}(z,s) = C_f C_a \Big\{ & \tilde{p}_{q\bar{q}}(-z) \Big[ \frac{2}{\epsilon} \Big( -2H_{-1,0} + H_{0,0} - \zeta_2 \Big) + \Big( 4H_{-1}\zeta_2 + 8H_{-1,-1,0} - 4H_{-1,0,0} \\
& - 8H_{0,-1,0} + 2H_{0,0,0} + 4H_{0,1,0} - 2\zeta_3 \Big) \Big] + \Big[ \frac{2}{\epsilon} \Big( 2(1-z) + (1+z)H_0 \Big) + \Big( -15(1-z) + 4(1+z)H_{-1,0} \\
& - (3+11z)H_0 + 4(1-z)H_{1,0} + 2(3-z)\zeta_2 \Big) \Big] \Big\} + C_f^2 \Big\{ \tilde{p}_{q\bar{q}}(-z) \Big[ \frac{4}{\epsilon} \Big( 2H_{-1,0} - H_{0,0} + \zeta_2 \Big) \\
& + \Big( -8H_{-1}\zeta_2 - 16H_{-1,-1,0} + 8H_{-1,0,0} + 16H_{0,-1,0} - 4H_{0,0,0} - 8H_{0,1,0} + 4\zeta_3 \Big) \Big] + \Big[ \frac{4}{\epsilon} \Big( -2(1-z) \\
& - (1+z)H_0 \Big) + \Big( 30(1-z) - 8(1+z)H_{-1,0} + (6+22z)H_0 - 8(1-z)H_{1,0} - 4(3-z)\zeta_2 \Big) \Big] \Big\} \\
+ C_f T_f \Big\{ \Big[ & \frac{1}{\epsilon^2} \Big( 4(1+z)H_0 + \frac{2(1-z)(4+7z+4z^2)}{3z} \Big) \\
& + \frac{1}{\epsilon} \Big( 4(1+z)H_{0,0} - \frac{2(3+3z+8z^2)}{3} H_0 - \frac{4(1-z)(16-11z+16z^2)}{9z} \Big) \\
& + \Big( 4(1+z)H_{0,0,0} - \frac{8(1-z)(2-z+2z^2)}{3z} H_{1,0} - \frac{2(3+3z+8z^2)}{3} H_{0,0} + 4(1+z)H_0\zeta_2 \\
& + \frac{4(21-12z+32z^2)}{9} H_0 - \frac{2(1-z)(4-11z+4z^2)}{3z} \zeta_2 + \frac{2(1-z)(136-71z+208z^2)}{27z} \Big) \Big] \Big\} .
\end{aligned} \tag{7.64}
$$

All other RR contributions are related by charge conjugation or flavor symmetry to these results as discussed in section 4.7. The only amplitude topology contributing to the last process is 7.1 (a) (or with different labeling alternatively (b)). To $\bar{q} \to q$, both amplitudes contribute. This gives rise to the two additional contribution proportional to $C_f C_a$ and $C_f^2$. To $q \to q + q\bar{q}$ again topology (b) will not contribute (with the labeling as above), but in addition amplitude (c) will contribute. In its calculation therefore no crossed topology (a×b) enters. For the other three splitting specified above, all three amplitude topologies contribute. Relevant for the kind of integrals and therefore the poles appearing is moreover, which propagators are populated by gluons and which by (anti)-quarks, because the first ones can give rise to additional denominators through the LC gauge propagators. The most singular contributions arise for the diagonal processes ($gg$ and $qq$), which contain poles up to $\alpha^{-2}$ and $\epsilon^{-4}$ with their combined power being at least $-4$.

The structure is mostly of the form discussed for the VR contribution in section 6.8. This is the fact for the proper appearance of the various color factors, the appearance of mass logarithms, which are completely contained in the first factors of eqns. (7.57, 7.58) and is different between the collinear and anti-collinear region, and the slightly different form of the results between the two regions, which is encoded in the $s$-dependence of the functions we presented on the last pages. Combined with the other contributions relevant at NNLO, these differences will eventually lead to a consistent cancellation of poles in $\alpha$ and the associated mass scale $v$. In the next section, we will discuss the relevant relations in detail. These will allow us to identify the matching kernels and anomaly coefficients introduced in chapter 3.

# 8 Refactorized Results

The results for the bare nTPDFs, which we presented in the last chapters, contain poles in the analytic regulator $\alpha$ and the dimensional regulator $\epsilon$. In section 3 we have discussed, how these poles are consistently removed. We will now first argue, how the consistency of the theory implies the $\alpha$ and $v$ independence of our result functions. After that we will make the relevant steps more explicit by expanding the corresponding equations and giving details about the calculational steps. Then we will give the explicit relations relevant for renormalization. After these discussions, we will list our main results, the renormalized functions $I_{i/k}$, $I'_{g/k}$, $F_{i\bar{\imath}}$, and the corresponding renormalization constants.

## 8.1 Refactorization

We now argue, why the cancellation of the analytic regulator $\alpha$ and its associated scale $v$ has to happen and how it gives rise to the collinear anomaly. We will follow arguments as in [5], but adjusted to the regulators, we apply.

### 8.1.1 Cancellation of $\alpha$ and v in the Final Result

The consistency of the theory implies that both, the analytic regulator $\alpha$ and the associated scale $v$, have to cancel in the result for the physical cross section. As the Wilson coefficient does not require this additional regularization and therefore does not depend on it, also the product of the other quantities has to be independent of it. This means in our case, the product $\mathcal{B}\bar{\mathcal{B}}$ cannot contain poles in $\alpha$ and allows setting $\alpha$ to 0. Then it also has to be independent of $v$. We have seen, however, that both $\mathcal{B}$ and $\bar{\mathcal{B}}$ contain both, poles in $\alpha$ and a dependence on $v$. So let us explore the implication of the constraints above.

We have argued in section 4.9 that we can express all $v$ dependence of the collinear function $\mathcal{B}$ solely in terms of the logarithm $L_c$ defined in eqn. (4.35), and correspondingly of the anti-collinear function $\bar{\mathcal{B}}$ solely in terms of the logarithm $L_a$ defined in eqn. (4.34). In addition to that $\mathcal{B}$ and $\bar{\mathcal{B}}$ depend on $z_1$ and $z_2$, respectively, as well as on $\alpha_s$ and $L_\perp$.

In their product $\alpha$ cancels and can be set to 0. Then it must be independent of $v$. The only possible way $L_c$ and $L_a$ can survive is by their $v$ independent combination

$$L_c - L_a = \log \frac{\bar{n}\cdot p\, n\cdot \bar{p}\, x_T^2}{4e^{-\gamma_e}} = \log \frac{q^2 x_T^2}{4e^{-\gamma_e}} - \log(z_1) - \log(z_2)\,. \tag{8.1}$$

This combination introduces the dependence on the hard scale $q^2$ through the logarithm

$$L_Q = \log \frac{x_T^2 q^2}{4e^{-2\gamma_e}} \,. \tag{8.2}$$

However, the dependence on this combination is highly constraint. Let us introduce

$$G = \log \left[\mathcal{B}_{i/j}(z_1, L_\perp, \alpha_s, L_c)\bar{\mathcal{B}}_{\bar{\imath}/k}(z_2, L_\perp, \alpha_s, L_a)\right]_{\alpha=0} ,$$

which has to be $v$ independent as argued above. Therefore,

$$0 \stackrel{!}{=} \frac{\partial}{\partial v} G \underset{\alpha=0}{=} \frac{\partial}{\partial v} \log \mathcal{B}_{i/j}(z_1, L_\perp, \alpha_s, L_c[v]) + \frac{\partial}{\partial v} \log \bar{\mathcal{B}}_{\bar{\imath}/k}(z_2, L_\perp, \alpha_s, L_a[v]) \,, \tag{8.3}$$

for all values of $v, z_1, z_2, i, j, k$, such that the parts in $\log \mathcal{B}_{i/j}$ and $\log \bar{\mathcal{B}}_{\bar{\imath}/k}$, which not cancel against each other or are dropped for $\alpha = 0$, must depend linearly on their argument $L_c$ and $L_a$, respectively, with coefficients that are equal, but bear opposite signs. Furthermore, the coefficients can depend on $\alpha_s$ $L_\perp$ and $i$ but not on $z_1$, $z_2$, $j$ or $k$. Then we can write

$$G = g_{i/j}(z_1, L_\perp, \alpha_s) + h_{\bar{\imath}/k}(z_2, L_\perp, \alpha_s) + f_{i\bar{\imath}}(L_\perp, \alpha_s)(L_c - L_a) \,, \tag{8.4}$$

where we introduced three new functions. Extracting the $z_1$ and $z_2$ dependence from $L_c - L_a$, this can then be rewritten as

$$G = \log B_{i/j}(z_1, L_\perp, \alpha_s) + \log B_{\bar{\imath}/k}(z_2, L_\perp, \alpha_s) + F_{i\bar{\imath}}(L_\perp, \alpha_s) \log \frac{q^2 x_T^2}{4e^{-2\gamma_e}} \,, \tag{8.5}$$

which is the refactorization formula. In the last equation we remove a remaining arbitrariness by demanding that the first two summands must be equal for same arguments and indices. (Note that $\bar{\bar{\imath}} = i$ and $\bar{g} = g$.) This is a very physical demand, as after the removal of the regulator, the collinear and anti-collinear contribution should be related by symmetry. Instead of choosing the constant factor $4e^{-2\gamma_e}$ in the logarithm containing $q^2$, we could have used any other constant. This would redistribute $z_{1,2}$ independent terms between the TPDFs and the term multiplied by $F$. If one wanted to parametrize this effect, one could introduce a massless scale $\eta$ in the logarithm containing $q^2$ and the two PDFs. As this effect is irrelevant for their product, we will not study this effect, however.

The main point for our purpose is that the consistency of the theory guarantees the cancellation of the poles in $\alpha$ and implies the refactorization formula, which can be written for the parton-to-parton nTPDFs as

$$\left[\mathcal{B}_{i/j}(z_1, x_T^2, \mu)\bar{\mathcal{B}}_{\bar{\imath}/k}(z_2, x_T^2, \mu)\right]_{q^2} \stackrel{\alpha=0}{=} e^{-L_Q F_{i\bar{\imath}}(x_T^2, \mu)} B_{i/j}(z_1, x_T^2, \mu) B_{\bar{\imath}/k}(z_2, x_T^2, \mu) \,. \tag{8.6}$$

In words, this equation means: The poles in $\alpha$ will cancel on the LHS. Once this is accomplished, the regulator is not needed anymore and is therefore dropped, such that the functions on the RHS are independent of it. They only depend on the partons $i, j, k$, the momentum fractions $z_1, z_2$, the transverse displacement $x_T^2$ and the renormalization scale $\mu$. Moreover, no distinction between the collinear and anti-collinear cases for the $B$ functions is needed. Instead of using the variables $\mu$ and $x_T^2$, one can change to $L_\perp$ and $\alpha_s(\mu)$. In the following, we will however leave that dependence implicit for the sake of brevity. We moreover suppress the label $^{(b)}$, that specifies all functions of this section as bare functions.

For the outgoing parton $i$ being a gluon eqn. (8.6) is a special case of eqn. (3.46), where we project on the first tensor structure of all TPDFs. By using different combination of projectors, we also obtain corresponding equations for $\mathcal{B}'\bar{\mathcal{B}}$, $\mathcal{B}\bar{\mathcal{B}}'$ and $\mathcal{B}'\bar{\mathcal{B}}'$. Of those, we will apply the first equation to extract $B'$. The other two equations are applied as checks. The discussion of eqn. (8.6) implies the equations for $B'$ by adding primes in the appropriate places.

We now expand both sides of eqn. (8.6) in $\frac{\alpha_s(\mu)}{4\pi}$ and compare the coefficients of each order. Note that $\alpha_s(\mu)$ is the renormalized coupling constant which is related to the bare constant via eqn. (4.27). The NNLO contribution thus contains a term of the form (4.28). The indices $^{(n)}$ in the expansion are as in eqn. (4.16).

### 8.1.2 At LO

At lowest order in the expansion, we obtain

$$\mathcal{B}^{(0)}_{i/j}(z_1)\bar{\mathcal{B}}^{(0)}_{\bar{\imath}/k}(z_2) = e^{-L_Q F^{(0)}_{i\bar{\imath}}} B^{(0)}_{i/j}(z_1) B^{(0)}_{\bar{\imath}/k}(z_2) \; . \tag{8.7}$$

With

$$\mathcal{B}^{(0)}_{i/j}(z_1) = \delta_{ij}\delta(1-z_1) \; , \;\; \bar{\mathcal{B}}^{(0)}_{\bar{\imath}/k}(z_2) = \delta_{\bar{\imath}k}\delta(1-z_2) \tag{8.8}$$

this implies

$$F^{(0)}_{i\bar{\imath}} = 0 \; , \tag{8.9}$$

$$B^{(0)}_{i/j}(z) = \delta_{ij}\delta(1-z) \; . \tag{8.10}$$

For the second gluon tensor structure we have

$$\mathcal{B}'^{(0)}_{i/j}(z_1) = 0 \; , \;\; \bar{\mathcal{B}}'^{(0)}_{\bar{\imath}/k}(z_2) = 0 \tag{8.11}$$

implying

$$B'^{(0)}_{i/j}(z) = 0 \; . \tag{8.12}$$

### 8.1.3 At NLO

For the first non trivial order, $\left(\frac{\alpha_s}{4\pi}\right)^1$, we have

$$\begin{aligned}\mathcal{B}^{(0)}_{i/j}(z_1)\bar{\mathcal{B}}^{(1)}_{\bar{\imath}/k}(z_2) + \mathcal{B}^{(1)}_{i/j}(z_1)\bar{\mathcal{B}}^{(0)}_{\bar{\imath}/k}(z_2) &= \\ B^{(0)}_{i/j}(z_1)B^{(1)}_{\bar{\imath}/k}(z_2) + B^{(1)}_{i/j}(z_1)B^{(0)}_{\bar{\imath}/k}(z_2) &- L_Q F^{(1)}_{i\bar{\imath}} B^{(0)}_{i/j}(z_1)B^{(0)}_{\bar{\imath}/k}(z_2)\,, \end{aligned} \tag{8.13}$$

where after the cancellation of poles in $\alpha$ on the LHS this regulator is set to 0 and the remaining logarithms containing the associated scale $v$ are combined as $L_c - L_a = L_Q - \log(z_1) - \log(z_2)$. Depending on the choice of partons $i, j, \bar{\imath}, k \in \{g, q_f, \bar{q}_f\}$ we distinguish three cases:

The off-diagonal-off-diagonal case, where $j \neq i \neq \bar{k}$, for which eqn. (8.13) collapses to $0 = 0$ and hence contains no information.

The diagonal-off-diagonal case, where either $j = i \neq \bar{k}$ or $\bar{\jmath} \neq \bar{\imath} = k$. For this case eqn. (8.13) collapses to

$$B^{(1)}_{i/j}(z) = \mathcal{B}^{(1)}_{i/j}(z) \;\text{ or }\; B^{(1)}_{\bar{\imath}/k}(z) = \bar{\mathcal{B}}^{(1)}_{\bar{\imath}/k}(z)\,,\ \text{respectively.} \tag{8.14}$$

The diagonal-diagonal case, where $j = i = \bar{k}$. Then eqn. (8.13) with eqns. (8.8, 8.10) does not simplify. This is also the only case, where poles in $\alpha$ appear. As can be seen from eqns. (5.19, 5.20), the signs of the terms with $\alpha^{-1}$ are exactly reversed between the collinear and anti-collinear nTPDF and are accompanied by a $\delta$-function, hence the $\alpha$ poles cancel on the LHS of eqn. (8.13). Moreover, expanding $\exp(\alpha L_{c,a})$, this directly implies that to $\alpha^0$ these logarithms only appear as single powers of $L_c - L_a = L_Q - \log(z_1) - \log(z_2)$. Terms suppressed by $\alpha$ are dropped which removes all remaining dependence on the scale $v$. That is, both the poles in $\alpha$ and the dependence on the associated scale $v$ drop out, leaving a dependence on $L_Q$. This is exactly the structure expected by general arguments leading to the collinear anomaly.

If the logarithms $\log(z_i)$, we just extracted, appear together with $\delta(1 - z_i)$, the corresponding terms vanish. Now, finding $B^{(1)}_{a/b}(z)$ and $F^{(1)}_{i\bar{\imath}}$ from the reformulated LHS is straight forward:

- $F^{(1)}_{i\bar{\imath}}$ is minus the prefactor of the sum of all terms proportional to $\delta(1-z_1)\delta(1-z_2)L_Q$.
- $B^{(1)}_{i/j}(z_1)$ is the prefactor of all terms proportional to $\delta(1 - z_2)$ but not to $\delta(1 - z_1)$, plus one-half times the terms proportional to $\delta(1 - z_1)\delta(1 - z_2)$ but not $L_Q$.
- $B^{(1)}_{\bar{\imath}/k}(z_2)$ is given by the corresponding terms with $z_1$ and $z_2$ interchanged. Note that by symmetry $B^{(1)}_{i/j}(z) = B^{(1)}_{\bar{\imath}/\bar{\jmath}}(z)$.

For $B'$ eqns. (8.11, 8.12) hold and therefore we obtain in analogy to eqn. (8.14)

$$B'^{(1)}_{i/j}(z) = \mathcal{B}'^{(1)}_{i/j}(z) \;\text{ and }\; B'^{(1)}_{\bar{\imath}/k}(z) = \bar{\mathcal{B}}'^{(1)}_{\bar{\imath}/k}(z)\,,\ \text{respectively.} \tag{8.15}$$

Which is, just as eqn. (8.14), free of poles in $\alpha$, such that the regulator can be dropped. Thus, no dependence on $v$ or $L_Q$ will arise.

### 8.1.4 At NNLO

Extracting the coefficients of $\left(\frac{\alpha_s}{4\pi}\right)^2$ in the expansion of eqn. (8.6) and solving for the three unknown functions, we receive

$$\begin{aligned}
&B^{(0)}_{i/j}(z_1)B^{(2)}_{\bar\imath/k}(z_2) + B^{(2)}_{i/j}(z_1)B^{(0)}_{\bar\imath/k}(z_2) - L_Q F^{(2)}_{i\bar\imath} B^{(0)}_{i/j}(z_1)B^{(0)}_{\bar\imath/k}(z_2) = \\
&\quad \mathcal{B}^{(2)}_{i/j}(z_1)\bar{\mathcal{B}}^{(0)}_{\bar\imath k}(z_2) + \mathcal{B}^{(0)}_{ij}(z_1)\bar{\mathcal{B}}^{(2)}_{\bar\imath/k}(z_2) + \mathcal{B}^{(1)}_{i/j}(z_1)\bar{\mathcal{B}}^{(1)}_{\bar\imath/k}(z_2) - B^{(1)}_{i/j}(z_1)B^{(1)}_{\bar\imath/k}(z_2) \\
&\quad + F^{(1)}_{i\bar\imath} L_Q \left(B^{(0)}_{ij}(z_1)B^{(1)}_{\bar\imath/k}(z_2) + B^{(1)}_{i/j}(z_1)B^{(0)}_{\bar\imath k}(z_2)\right) - \tfrac{1}{2}(F^{(1)}_{i\bar\imath} L_Q)^2 B^{(0)}_{ij}(z_1)B^{(0)}_{\bar\imath k}(z_2)\,.
\end{aligned} \tag{8.16}$$

Again the poles in $\alpha$ cancel between the terms including $\mathcal{B}$ and that regulator is dropped there after. In addition to that, the $v$ dependence via $L_c$ and $L_a$ cancels, but gives rise to $L_Q$. We will give some additional remarks to these points later. The RHS contains the product $\mathcal{B}^{(1)}_{i/j}(z_1)\bar{\mathcal{B}}^{(1)}_{\bar\imath/k}(z_2)$, where each individual factor contains poles in both regulators, which were found to be up to $\epsilon^{-3}$, $\alpha^{-1}$ for the diagonal cases and $\epsilon^{-2}$, $\alpha^0$ for the off-diagonal cases. Therefore, $\mathcal{B}^{(1)}$ has to be determined to the corresponding order in $\alpha$ and $\epsilon$. Correspondingly $B^{(1)}$ has to be determined to sufficient order in $\epsilon$ to properly determine the term $B^{(1)}_{i/j}(z_1)B^{(1)}_{\bar\imath/k}(z_2)$. $\mathcal{B}^{(2)}_{i/j}$ contains the corresponding double-real and the virtual-real terms. In addition to that it also contains the contribution implied by the NLO result and eqn. (4.28), i.e.

$$\mathcal{B}^{(2)}_{i/j} = \mathcal{B}^{(RR)}_{i/j} + \mathcal{B}^{(VR)}_{i/j} + 2Z^{(1)}_{g_s}\mathcal{B}^{(1)}_{i/j}\,. \tag{8.17}$$

An analog statement holds for the anti-collinear function.

Using eqns. (8.8, 8.10), in some cases this equation simplifies: For the off-diagonal-off-diagonal case, it reduces to $\mathcal{B}^{(1)}_{i/j}(z_1)\bar{\mathcal{B}}^{(1)}_{\bar\imath/k}(z_2) - B^{(1)}_{i/j}(z_1)B^{(1)}_{\bar\imath/k}(z_2) = 0$ which is trivially fulfilled due to eqn. (8.14).

For the diagonal-off-diagonal case, we receive for $j = i \neq \bar k$

$$\begin{aligned}
\delta(1-z_1)B^{(2)}_{\bar\imath/k}(z_2) =& \delta(1-z_1)\bar{\mathcal{B}}^{(2)}_{\bar\imath/k}(z_2) + \mathcal{B}^{(1)}_{i/j}(z_1)\bar{\mathcal{B}}^{(1)}_{\bar\imath/k}(z_2) \\
&- B^{(1)}_{i/j}(z_1)B^{(1)}_{\bar\imath/k}(z_2) + F^{(1)}_{i\bar\imath} L_Q \delta(1-z_1)B^{(1)}_{\bar\imath/k}(z_2)\,.
\end{aligned} \tag{8.18}$$

For $\bar\jmath \neq \bar\imath = k$, an analogous relation holds. From the explicit results provided in the last section, one can also see explicitly, how the poles in $\alpha$ cancel between the first and the second term on the RHS due to the inverted signs between them and how these terms give rise to an $L_Q$ dependence when the scale $v$ is removed. The $L_Q$ dependence is then canceled by the last term of the RHS and the corresponding results for $B^{(2)}$ are easily extracted. For the special cases ($q/\bar q$, $q/q'$, $q/\bar q'$), where in addition to $\mathcal{B}^{(0)}_{\bar\imath/k}(z)$ also $\mathcal{B}^{(1)}_{\bar\imath/k}(z)$ vanishes, eqn. (8.18) collapses to $B^{(2)}_{a/b}(z) = \mathcal{B}^{(2)}_{a/b}(z)$.

For the diagonal-diagonal case, all terms in eqn. (8.16) contribute. The poles in $\alpha$ and the $v$-dependence cancel between the first three terms on the RHS, most of them between the first two terms. Just as in the earlier discussions, the poles in $\alpha$ induce a dependence on $L_c - L_a = L_Q - \log(z_1 z_2)$ once the corresponding exponentials are expanded. The $\mathcal{O}(\alpha)$ terms and with them the remaining $v$ dependence are dropped. We then can extract the functions on the LHS of eqn. (8.16) analogously as described for the corresponding NLO case: $F^{(2)}_{i\bar{\imath}}$ as minus the prefactor of the sum of all terms proportional to $\delta(1-z_1)\delta(1-z_2)L_Q$; $B^{(2)}_{i/j}(z_1)$ as the prefactor of all terms proportional to $\delta(1-z_2)$ but not to $\delta(1-z_1)$, plus one-half times the terms proportional to $\delta(1-z_1)\delta(1-z_2)$ but not $L_Q$. $B^{(2)}_{\bar{\imath}/k}(z_2)$ by the corresponding terms with $z_1$ and $z_2$ interchanged.

To obtain $B'^{(2)}_{g/j}$, we again consider the analog of eqn. (8.16) with a prime on either of the (n)TPDFs. Due to eqns. (8.11, 8.12), we then obtain equations analog to (8.18), but with a prime on the $z_2$ dependent functions as well as an alternative equation including $\bar{\mathcal{B}}'^{(2)}_{g/k}$. The cancellation of poles in $\alpha$, the dropping out of the $v$ dependence for $\alpha = 0$, the arising of $L_Q$ and its cancellation between the different terms as well as the extraction of $B'^{(2)}_{g/j}$ works in full analogy as discussed for (8.18). Unfortunately, for the time being there are some unresolved issues with our NNLO results for the second tensor structure. We therefore do not provide them here.

Let us summarize. In this section we have seen, how the bare functions $F$ and $B$ are obtained up to NNLO and $B'$ up to NLO in the renormalized coupling. We have seen, how the poles in $\alpha$ cancel and how the logarithms of the associated mass scale give rise to the collinear anomaly in form of a logarithm of the hard scale. Before we will provide the final results, we discuss the remaining operator renormalization in the next section.

## 8.2 Renormalization

The renormalization is performed in the $\overline{\text{MS}}$-scheme. In the coefficients $\mathcal{B}^{(n)}$ and $\bar{\mathcal{B}}^{(n)}$, we already included the effects from moving from the bare to the renormalized coupling. This was effectively done by eqn. (8.17). In this section, we will apply the still required operator renormalization of our functions.

Before starting the renormalization, we recall from eqns. (3.19, 3.53), that up to corrections suppressed by $\Lambda^2_{\text{QCD}} x_T^2$,

$$B_{i/j}(z) = \sum_k I_{i/k}(z) \otimes \phi_{k/j}(z) \tag{8.19}$$

and correspondingly for $B'$. In this equation and in the following, we use the short notation (B.34) for the Mellin convolution. We moreover suppress the arguments $\mu$ and $x_T^2$. Since in dimensional regularization we found the simple result

$$\phi^{(b)}_{i/j}(z) \equiv \delta_{i/j}\delta(1-z)$$

for the bare PDF, the results of the bare TPDFs is identical to the results of the bare matching kernels, i.e.

$$B^{(b)}_{i/j}(z) = I^{(b)}_{i/j}(z)\,, \tag{8.20}$$
$$B'^{(b)}_{i/j}(z) = I'^{(b)}_{i/j}(z)\,. \tag{8.21}$$

From the results of the bare $F^{(b)}$, $I^{(b)}$ and $I'^{(b)}$ functions obtained in this sense in the way explained in the last section as well as from the results of the bare PDF $\phi^{(b)}$, we will now extract the corresponding renormalized functions.

To this end, we will use the operator renormalization eqns. (3.24, 3.48, 3.23, 3.47) for $F$, $B$ and $B'$, which are

$$F^{(b)}_{i\bar{\imath}} = F^{(r)}_{i\bar{\imath}} + Z^F_i\,, \tag{8.22}$$
$$B^{(b)}_{i/j}(z) = Z^B_i B^{(r)}_{i/j}(z)\,, \tag{8.23}$$
$$B'^{(b)}_{i/j}(z) = Z^B_i B'^{(r)}_{i/j}(z) \tag{8.24}$$

and the operator renormalization equation for $\phi$,

$$\phi^{(b)}_{i/j}(z) = \sum_k Z^{\phi}_{i/k}(z) \otimes \phi^{(r)}_{k/j}(z) = \delta_{i/j}\delta(1-z)\,. \tag{8.25}$$

The last equality states our results for the bare PDFs in dimensional regularization with $\epsilon = \epsilon_{UV} = \epsilon_{IR}$.

As we work in the $\overline{\mathrm{MS}}$-scheme, in the last four equations, all UV poles on the RHS are contained in $Z^F_i$, $Z^B_i - 1$ and $Z^{\phi}_{i/k}(z) - \delta_{ij}\delta(1-z)$, while the renormalized functions are free of such poles. Therefore, the last equation implies that $\phi^{(r)}_{i/j}(z) - \phi^{(r,0)}_{i/j}(z)$ purely consists of IR-poles. We also observe, $Z^{\phi}$ and $\phi^{(r)}$ are Mellin inverses, i.e. $\phi^{(r)}(z) = (Z^{\phi})^{-1}(z)$ at $\epsilon_{UV} = \epsilon_{IR}$, in the sense of eqn. (8.25).

Considering eqn. (8.23), we can identify the LHS as $I^{(b)}_{i/j}(z)$ and apply eqn. (8.19) for $B^{(r)}_{i/j}(z)$ on the RHS. We then find the renormalization equation for the matching kernel

$$I^{(b)}_{i/j}(z) = Z^B_i \sum_k I^{(r)}_{i/k}(z) \otimes \phi^{(r)}_{k/j}(z)\,. \tag{8.26}$$

An analogous equation holds for $I'$. Given our considerations of the last paragraph and especially $(Z^{\phi})^{-1}(z) = \phi^{(r)}(z)$, this equation is consistent with the fact that eqn. (8.19) has to hold for both, the renormalized and bare functions.

One can understand eqn. (8.26) as follows. The left hand side, which is equal to $B^{(b)}$, contains both, UV and IR poles. The IR poles of $B$, however, are the same as for $\phi$. Thus, the RHS consists of the $Z$ factor containing all UV poles, the PDF containing all IR poles and the matching kernel $I$ which is free of both kinds of poles.

We will now discuss the order by order implications of the last equation. At LO no renormalization is needed and we simply have

$$Z_i^{B(0)} = 1\,, \tag{8.27}$$

$$I'^{(b,0)}_{i/j}(z),\, I'^{(r,0)}_{i/j} = 0\,, \tag{8.28}$$

$$I^{(b,0)}_{i/j}(z),\, I^{(r,0)}_{i/j}(z),\, \phi^{(b,0)}_{i/j}(z),\, \phi^{(r,0)}_{i/j}(z) = \delta_{ij}\delta(1-z)\;. \tag{8.29}$$

At NLO we then obtain

$$I^{(r,1)}_{i/j}(z) + Z_i^{B(1)}\delta_{i/j}\delta(1-z) + \phi^{(r,1)}_{i/j}(z) = I^{(b,1)}_{i/j}(z)\;. \tag{8.30}$$

This means, all terms from the RHS without poles in $\epsilon$ are part of $I^{(r,1)}_{i/j}(z)$ and all terms with poles in $\epsilon$ which do not contain $\delta(1-z)$ belong to $\phi^{(r,1)}_{i/j}(z)$. For $i \neq j$ also all potential $\epsilon$-poles coming with $\delta(1-z)$ would be part of $\phi^{(r,1)}_{i/j}(z)$. However, such terms do not appear for them. For $i = j$ the ambiguity of distributing the $\epsilon$-pole terms proportional to $\delta(1-z)$ between $\phi^{(r,1)}_{i/j}(z)$ and $Z_i^{B(1)}\delta(1-z)$ is resolved by constraining the integrals containing $\phi^{(r,1)}_{i/j}(z)$ or $z\phi^{(r,1)}_{i/j}(z)$, respectively. These constraints are directly related to the constraints on the integrals of the DGLAP splitting kernels,

$$\begin{aligned} &\int_0^1 \mathrm{d}z \big[P_{qq}(z) - P_{q\bar{q}}(z)\big] = 0\,, \\ &\int_0^1 \mathrm{d}z\; z\big[P_{qq}(z) + P_{gq}(z)\big] = 0\,, \quad \text{and} \\ &\int_0^1 \mathrm{d}z\; z\big[P_{gg}(z) + 2N_f P_{qg}(z)\big] = 0\,, \end{aligned} \tag{8.31}$$

which follow from momentum and quark number conservation. In Appendix C.3, we state relations between the renormalized PDFs and splitting kernels. From them, one easily deduces the constraints for $\phi^{(r,n)}_{i/j}(z)$. Since the endpoint contributions of the off-diagonal terms are fixed from eqn. (8.30), these equations fix the endpoint contributions of the diagonal terms. From the results obtained for $\phi^{(r,n)}_{i/j}(z)$, one can then straightforwardly obtain the corresponding splitting kernels.

At NNLO eqn. (8.26), solved for the unknown second order functions, reads

$$\begin{aligned} &I^{(r,2)}_{i/j}(z) + Z_i^{B(2)}\delta_{i/j}\delta(1-z) + \phi^{(r,2)}_{i/j}(z) = \\ &\qquad I^{(b,2)}_{i/j}(z) - \sum_k I^{(r,1)}_{i/k}(z)\otimes\phi^{(r,1)}_{k/j}(z) - Z_i^{B(1)}\big(I^{(r,1)}_{i/j}(z) + \phi^{(r,1)}_{i/j}(z)\big)\,. \end{aligned} \tag{8.32}$$

Note that for the RHS, $I^{(r,1)}$ has to be determined up to its $\epsilon^2$ terms. The distribution of the terms between the three functions on the LHS is done as at NLO. One additional complication is the second term on the RHS with the convolution, especially since both functions contain distributions and the integral does not cover the whole support of these distributions. We give more detail about such integrals in section B.5.

From the results obtained for $\phi^{(r)}$ and eqn. (8.25), also $Z^\phi$ can be extracted order by order. It is given by

$$Z^{\phi(1)}_{i/j}(z) = -\phi^{(r,1)}_{i/j}(z)\,, \tag{8.33}$$

$$Z^{\phi(2)}_{i/j}(z) = -\phi^{(r,2)}_{i/j}(z) - \sum_k Z^{\phi(1)}_{i/k}(z) \otimes \phi^{(r,1)}_{k/j}(z)\,. \tag{8.34}$$

In addition to the functions $I$, we also have to renormalize the functions $I'$. The basic relation is again (8.26), this time with $I'$ in place of $I$. Due to eqn. (8.28), the resulting expanded expressions are simpler. At NLO, we simply obtain

$$I'^{(r,1)}_{g/j}(z) = I'^{(b,1)}_{g/j}(z)\,, \tag{8.35}$$

i.e. there are no poles in $I'$ at this order.

At NNLO, we have

$$I'^{(r,2)}_{g/j}(z) = I'^{(b,2)}_{g/j}(z) - \sum_k I'^{(r,1)}_{g/k}(z) \otimes \phi^{(r,1)}_{k/j}(z) - Z^{B(1)}_g I'^{(r,1)}_{g/j}(z)\,, \tag{8.36}$$

where we can use the NLO expressions for $Z^B_i$ and $\phi$ which were obtained from eqn. (8.30).

Despite $I$ and $I'$, we also renormalize $F$. The relation (8.22) is very simple: At each order in $\alpha_s$, all poles in $\epsilon$ are absorbed into $Z^F$, while the finite terms give rise to $F^{(r)}$.

Applying first the refactorization and then the renormalization in the way discussed in this and the last section, we obtain the final results for $I_{i/j}$, $I'_{g/j}$, $F_{i\bar{\imath}}$, $\phi_{i/j}$ and the related renormalization constants. The explicit labels $^{(r)}$ will be dropped from now on. We will list the NLO results for $Z^F_i$, $Z^B_i$, $F_{i\bar{\imath}}$, $I_{i/j}$ and $I'_{i/j}$ in the next section and then continue with our main results, the NNLO results of the first four of them. The results for $\phi_{i/j}$ and its renormalization factor, are directly related to well known DGLAP kernels, which are listed in Appendix C.3. Hence, we do not provide them explicitly.

## 8.3 NLO results

The last section explained explicitly, how - starting from the bare results of the (nT)PDFs presented in sections 5.6, 6.8, and 7.9 - we can extract the refactorized $I$ and $F$ functions and renormalize them. We now give the final results for those functions. We thereby suppress the label $^{(r)}$ and start with the NLO results in this section.

### Renormalization Constants

The NLO renormalization constants for the anomaly coefficients are given by

$$Z_i^{F(1)} = \frac{\Gamma_0^i}{\epsilon} \,. \tag{8.37}$$

The renormalization constants for the TPDFs take the form

$$Z_i^{B(1)}(L_\perp) = \frac{\Gamma_0^i}{2\epsilon^2} + \frac{\Gamma_0^i\, L_\perp - 2\gamma_0^i}{2\epsilon} \,. \tag{8.38}$$

The anomalous dimensions are listed in Appendix C.1.

### Anomaly Coefficients

The NLO anomaly coefficients are obtained as

$$\frac{F_{q\bar{q}}^{(1)}(L_\perp)}{C_f} = \frac{F_{gg}^{(1)}(L_\perp)}{C_a} = 4L_\perp \,. \tag{8.39}$$

### Matching Kernels

In contrast to the other functions, the renormalized NLO matching kernels are needed up to $\epsilon^2$ for our later purpose, since in the extraction of the NNLO matching kernels in eqn. (8.32), they are multiplied by the renormalization factors $Z^{B(1)}$ containing double poles in $\epsilon$. Up to this order they result into

$$\begin{aligned}
I_{g/g}^{(1)}(z, L_\perp) = {} & C_a\Big[-(\zeta_2 - L_\perp^2)\delta(1-z) - 4L_\perp \tilde{p}_{gg}(z)\Big] + \epsilon\, C_a\Big[-2(\zeta_2 + L_\perp^2)\tilde{p}_{gg}(z) \\
& - \frac{3\zeta_2 L_\perp + 4\zeta_3 - L_\perp^3}{3}\delta(1-z)\Big] + \epsilon^2 C_a\Big[\frac{-2L_\perp^3 - 6L_\perp\zeta_2 - 4\zeta_3}{3}\tilde{p}_{gg}(z) \\
& + \delta(1-z)\left(\frac{L_\perp^4}{12} - \frac{L_\perp^2\zeta_2}{2} - \frac{4L_\perp\zeta_3}{3} - \frac{27\zeta_4}{8}\right)\Big] \,, \\
I_{g/q}^{(1)}(z, L_\perp) = {} & C_f\Big[2z - 2L_\perp \tilde{p}_{gq}(z)\Big] + \epsilon\, C_f\Big[-(\zeta_2 + L_\perp^2)\tilde{p}_{gq}(z) + 2L_\perp z\Big] \\
& + \epsilon^2 C_f\left[\frac{-L_\perp^3 - 3L_\perp\zeta_2 - 2\zeta_3}{3}\tilde{p}_{gq}(z) + z(L_\perp^2 + \zeta_2)\right] \,,
\end{aligned} \tag{8.40}$$

$$
\begin{aligned}
I^{(1)}_{q/g}(z,L_\perp) =& T_f\Big[2-2(1+L_\perp)\tilde{p}_{qg}(z)\Big] + \epsilon\, T_f\Big[2(1+L_\perp)-(2+\zeta_2+2L_\perp+L_\perp^2)\tilde{p}_{qg}(z)\Big] \\
&+ \epsilon^2 T_f\left[\frac{-(L_\perp+1)^3-3(L_\perp+1)(\zeta_2+1)-2(\zeta_3+1)}{3}\tilde{p}_{qg}(z)+(L_\perp+1)^2+\zeta_2+1\right]\,, \\
I^{(1)}_{q/q}(z,L_\perp) =& C_f\Big[2(1-z)-(\zeta_2-L_\perp^2)\delta(1-z)-2L_\perp\tilde{p}_{qq}(z)\Big] \\
&+ \epsilon\, C_f\Big[-\frac{3\zeta_2 L_\perp+4\zeta_3-L_\perp^3}{3}\delta(1-z)-(\zeta_2+L_\perp^2)\tilde{p}_{qq}(z)+2L_\perp(1-z)\Big] \\
&+ \epsilon^2 C_f\Big[\frac{-L_\perp^3-3L_\perp\zeta_2-2\zeta_3}{3}\tilde{p}_{qq}(z)+\delta(1-z)\left(\frac{L_\perp^4}{12}-\frac{L_\perp^2\zeta_2}{2}-\frac{4L_\perp\zeta_3}{3}-\frac{27\zeta_4}{8}\right) \\
&\qquad + (1-z)(L_\perp^2+\zeta_2)\Big]\,.
\end{aligned}
$$

For the second tensor structure of the gluon kernels, we obtain

$$
\frac{I'^{(1)}_{g/g}(z,L_\perp)}{C_a} = \frac{I'^{(1)}_{g/q}(z,L_\perp)}{C_f} = 4\,\frac{1-z}{z}\left(1+\epsilon L_\perp+\epsilon^2\frac{\zeta_2+L_\perp^2}{2}\right). \tag{8.41}
$$

## 8.4 NNLO results

Just as the NLO results, the final NNLO results can be obtained by the steps described in sections 8.1 and 8.2.

### 8.4.1 Renormalization Constants

The renormalization constants are given by

$$Z_i^{F(2)} = -\frac{\beta_0 \Gamma_0^i}{2\epsilon^2} + \frac{\Gamma_1^i}{2\epsilon}\,, \tag{8.42}$$

for the anomaly coefficient, and for the TPDFs it can be written as

$$\begin{aligned} Z_i^{B(2)}(L_\perp) = &\frac{(\Gamma_0^i)^2}{8\epsilon^4} + \frac{\Gamma_0^i}{4\epsilon^3}\left[\Gamma_0^i\, L_\perp - 2\gamma_0^i - \frac{3}{2}\,\beta_0\right] \\ &+ \frac{1}{8\epsilon^2}\left[\Gamma_1^i + \left(\Gamma_0^i\, L_\perp - 2\gamma_0^i\right)^2 - 2\beta_0\left(\Gamma_0^i\, L_\perp - 2\gamma_0^i\right)\right] + \frac{1}{4\epsilon}\left(\Gamma_1^i\, L_\perp - 2\gamma_1^i\right), \end{aligned} \tag{8.43}$$

with the constants as listed in Appendix C.1 and C.2.

### 8.4.2 Anomaly Coefficients

The NNLO anomaly coefficients result into

$$\begin{aligned} \frac{F_{q\bar{q}}^{(2)}(L_\perp)}{C_f} = \frac{F_{gg}^{(2)}(L_\perp)}{C_a} = &C_a\left[\frac{808}{27} - 28\zeta_3 + \frac{268}{9}L_\perp - 8\zeta_2 L_\perp + \frac{22}{3}L_\perp^2\right] \\ &- T_f N_f\left[\frac{224}{27} + \frac{80}{9}L_\perp + \frac{8}{3}L_\perp^2\right]. \end{aligned} \tag{8.44}$$

### 8.4.3 Matching Kernels

Just as the results of the bare nTPDFs, we will present the results of the matching kernels in terms of harmonic polylogarithms $H_{\vec{a}_n} \equiv H_{\vec{a}_n}(z)$ up to weight $n = 3$, $\zeta$-values up to weight 4 and functions $\tilde{p}_{ij}$ related to the lowest order DGLAP splitting kernels $P_{ij}^{(0)}$ by removing an overall factor and the $\delta$-function. These functions are discussed in sections B.2 and C.3, respectively.

To reduce the results to a compact size, in addition to that, we will only give their scale independent part, i.e. their results at $L_\perp = 0$ which is obtained for $\mu = \mu_x \equiv \frac{2e^{-\gamma_e}}{x_T}$. The corresponding expressions at $\mu \neq \mu_x$, containing powers of $L_\perp$, can straightforwardly be obtained from these expressions by solving the RGE eqns. (3.31,3.54) of $I_{i/j}$. To this end, the QCD $\beta$-function, the cusp, quark and gluon anomalous dimensions as well as the DGLAP splitting kernels are needed to $\alpha_s^2$. Their well known expressions are provided or referred to in Appendix C.1 and C.3.

The scale independent part of the NNLO gluon-to-gluon kernel is given by

$$
\begin{aligned}
I^{(2)}_{g/g}(z, L_\perp{=}0) &= C_a^2 \bigg\{ \delta(1-z)\bigg[\frac{25}{4}\zeta_4 - \frac{77}{9}\zeta_3 - \frac{67}{6}\zeta_2 + \frac{1214}{81}\bigg] \\
&+ \tilde{p}_{gg}(z)\bigg[ -4H_{0,0,0} + 8H_{0,1,0} + 8H_{0,1,1} - 8H_{1,0,0} + 8H_{1,0,1} + 8H_{1,1,0} + 52\zeta_3 - \frac{808}{27}\bigg] \\
&+ \tilde{p}_{gg}(-z)\bigg[ -16H_{-1,-1,0} + 8H_{-1,0,0} + 16H_{0,-1,0} - 4H_{0,0,0} - 8H_{0,1,0} - 8H_{-1}\zeta_2 + 4\zeta_3\bigg] \\
&+ \bigg[ -16(1+z)H_{0,0,0} + \frac{2(25-11z+44z^2)}{3}H_{0,0} + \frac{8(1-z)(11-z+11z^2)}{3z}\big(H_{1,0}+\zeta_2\big) \\
&\quad - \frac{2z}{3}H_1 - \frac{(701+149z+536z^2)}{9}H_0 + \frac{4(-196+174z-186z^2+211z^3)}{9z}\bigg]\bigg\} \\
&+ C_a T_f N_f \bigg\{ \delta(1-z)\bigg[\frac{28}{9}\zeta_3 + \frac{10}{3}\zeta_2 - \frac{328}{81}\bigg] + \frac{224}{27}\tilde{p}_{gg}(z) \\
&+ \bigg[\frac{8(1+z)}{3}H_{0,0} + \frac{4z}{3}H_1 + \frac{4(13+10z)}{9}H_0 - \frac{4(-65+54z-54z^2+83z^3)}{27z}\bigg]\bigg\} \\
&+ C_f T_f N_f \bigg\{ 8(1+z)H_{0,0,0} + 4(3+z)H_{0,0} + 24(1+z)H_0 - \frac{8(1-z)(1-23z+z^2)}{3z}\bigg\}.
\end{aligned} \tag{8.45}
$$

The quark-to-gluon kernel is obtained as

$$
\begin{aligned}
I^{(2)}_{g/q}(z, L_\perp{=}0) &= C_f C_a \bigg\{ \tilde{p}_{gq}(z)\bigg[4H_{1,1,1} + 4H_{0,1,1} + 4H_{1,0,1} + 4H_{1,1,0} + 8H_{0,1,0} \\
&\quad - 4H_{1,0,0} - \frac{22}{3}H_{1,1} + \frac{44}{3}\big(H_{1,0}+\zeta_2\big) + \frac{152}{9}H_1 + 24\zeta_3 - \frac{1580}{27}\bigg] \\
&+ \tilde{p}_{gq}(-z)\bigg[ -8H_{-1,-1,0} + 4H_{-1,0,0} + 8H_{0,-1,0} - 4H_{-1}\zeta_2\bigg] \\
&+ \bigg[16H_{0,1,0} - 4(2+z)H_{0,0,0} + 4zH_{-1,0} + 4zH_{0,1} + 4zH_{1,1} - \frac{8(1+z+2z^2)}{3}H_{1,0} \\
&\quad + \frac{2(36+9z+8z^2)}{3}H_{0,0} - \frac{22z}{3}H_1 - \frac{2(249-6z+88z^2)}{9}H_0 \\
&\quad - 8\zeta_3 - \frac{2(4+13z+8z^2)}{3}\zeta_2 + \frac{4(1+127z+152z^2)}{27}\bigg]\bigg\} \\
&+ C_f^2 \bigg\{ \tilde{p}_{gq}(z)\bigg[ -4H_{1,1,1} + 6H_{1,1} - 16H_1\bigg] \\
&+ \bigg[2(2-z)H_{0,0,0} - (4+3z)H_{0,0} - 4zH_{1,1} + 6zH_1 - 5(3-z)H_0 + (10-z)\bigg]\bigg\} \\
&+ C_f T_f N_f \bigg\{ \tilde{p}_{gq}(z)\bigg[\frac{8}{3}H_{1,1} - \frac{40}{9}H_1 + \frac{224}{27}\bigg] + \bigg[\frac{8z}{3}H_1 - \frac{40z}{9}\bigg]\bigg\},
\end{aligned} \tag{8.46}
$$

while the gluon-to-quark kernel results in

$$
\begin{aligned}
I^{(2)}_{q/g}(z, L_\perp{=}0) = C_a T_f \Big\{ &\tilde{p}_{qg}(z)\Big[4H_{1,0,1} + 4H_{1,1,0} - 4H_{1,1,1} + 4H_{1,1} - \frac{44}{3}H_{0,0} \\
&+ \frac{44}{3}\big(H_{1,0} + \zeta_2\big) + \frac{136}{9}H_0 + 4H_1 - \frac{298}{27}\Big] \\
&+ \tilde{p}_{qg}(-z)\Big[-8H_{-1,-1,0} + 4H_{-1,0,0} + 8H_{0,-1,0} + 4H_{-1,0} - 4H_{-1}\zeta_2\Big] \\
&+ \Big[-16zH_{0,1,0} + 4(1+2z)H_{0,0,0} + \frac{2(19-32z)}{3}H_{0,0} - 4H_{-1,0} - 4H_{1,1} - \frac{4(13-38z)}{9}H_0 \\
&- \frac{4(4+5z+2z^2)}{3z}\big(H_{1,0}+\zeta_2\big) + 2(-2+z)H_1 + 8z\zeta_3 + 8z\zeta_2 + \frac{2(172-166z+89z^2)}{27z}\Big]\Big\} \\
+ C_f T_f \Big\{ &\tilde{p}_{qg}(z)\Big[4H_{1,1,1} - 4H_{1,0,0} + 4H_{0,1,1} - 4H_{0,0,0} - 4H_{1,1} - 4H_{1,0} - 4H_{0,1} - 4H_{0,0} \\
&- 4H_1 - 4H_0 + 28\zeta_3 + 6\zeta_2 - 36\Big] + \Big[2(1-2z)H_{0,0,0} + (5+4z)H_{0,0} + 4H_{0,1} + 4H_{1,0} \\
&+ 4H_{1,1} + 2(2-z)H_1 + (12+7z)H_0 - 6\zeta_2 + (23+3z)\Big]\Big\}\,.
\end{aligned}
\tag{8.47}
$$

The matching kernel for a quark evolving to a quark of the same flavor is

$$
\begin{aligned}
I^{(2)}_{q/q}(z, L_\perp{=}0) = C_f C_a \Big\{ &\delta(1-z)\Big[5\zeta_4 - \frac{77}{9}\zeta_3 - \frac{67}{6}\zeta_2 + \frac{1214}{81}\Big] \\
&+ \tilde{p}_{qq}(z)\Big[-2H_{0,0,0} - 4H_{0,1,0} - 4H_{1,0,1} - 4H_{1,1,0} - \frac{11}{3}H_{0,0} - \frac{76}{9}H_0 + 2\zeta_3 - \frac{404}{27}\Big] \\
&+ \Big[-4(1-z)H_{1,0} - 4zH_{0,0} - 2zH_1 + 2(1+5z)H_0 - 6(1-z)\zeta_2 + \frac{44}{3}(1-z)\Big]\Big\} \\
+ C_f^2 \Big\{ &\frac{5}{4}\zeta_4\,\delta(1-z) \\
&+ \tilde{p}_{qq}(z)\Big[8H_{0,1,0} + 4H_{0,1,1} - 4H_{1,0,0} + 8H_{1,0,1} + 8H_{1,1,0} + 3H_{0,0} + 8H_0 + 24\zeta_3\Big] \\
&+ \Big[2(1+z)H_{0,0,0} + (3+7z)H_{0,0} + 4(1-z)H_{0,1} + 12(1-z)H_{1,0} + 2zH_1 \\
&+ 2(1-12z)H_0 + 6(1-z)\zeta_2 - 22(1-z)\Big]\Big\} \\
+ C_f T_f N_f \Big\{ &\delta(1-z)\Big[\frac{28}{9}\zeta_3 + \frac{10}{3}\zeta_2 - \frac{328}{81}\Big] + \tilde{p}_{qq}(z)\Big[\frac{4}{3}H_{0,0} + \frac{20}{9}H_0 + \frac{112}{27}\Big] - \frac{4}{3}(1-z)\Big\}\,.
\end{aligned}
\tag{8.48}
$$

For a quark evolving to a quark (or anti-quark) of different flavor, it is instead given by

$$I^{(2)}_{q'/q}(z, L_\perp{=}0) = C_f T_f \bigg\{ 4(1+z)H_{0,0,0} - \frac{2(3+3z+8z^2)}{3} H_{0,0} - \frac{8(1-z)(2-z+2z^2)}{3z} \big(H_{1,0} + \zeta_2\big) + \frac{4(21-30z+32z^2)}{9} H_0 + \frac{2(1-z)(172-143z+136z^2)}{27z} \bigg\}. \tag{8.49}$$

For a quark evolving to an anti-quark of the same flavor, the matching kernel is

$$\begin{aligned} I^{(2)}_{\bar{q}/q}(z, L_\perp{=}0) &= \big(C_f C_a - 2C_f^2\big) \bigg\{ \tilde{p}_{qq}(-z) \Big[ 8H_{-1,-1,0} - 4H_{-1,0,0} - 8H_{0,-1,0} + 4H_{0,1,0} \\ &\quad + 2H_{0,0,0} + 4H_{-1}\zeta_2 - 2\zeta_3 \Big] \\ &\quad + \Big[ 4(1-z)H_{1,0} + 4(1+z)H_{-1,0} - (3+11z)H_0 + 2(3-z)\zeta_2 - 15(1-z) \Big] \bigg\} \\ &\quad + C_f T_f \bigg\{ 4(1+z)H_{0,0,0} - \frac{2(3+3z+8z^2)}{3} H_{0,0} - \frac{8(1-z)(2-z+2z^2)}{3z} \big(H_{1,0} + \zeta_2\big) \\ &\quad + \frac{4(21-30z+32z^2)}{9} H_0 + \frac{2(1-z)(172-143z+136z^2)}{27z} \bigg\}. \end{aligned} \tag{8.50}$$

In chapter 3, we discussed, the relevance and process-independence of the matching kernels and anomaly coefficients. In particular, we have seen, how the $I_{i/j}$ functions determined to $\alpha_s^2$ form a process-independent piece required for N$^3$LL $q_T$ resummation for a wide range of processes at hadron colliders, in which a color-neutral final state with high invariant mass and small transverse momentum is produced. We do not provide the results for $I'^{(2)}_{g/j}$ in this work. As discussed earlier, for most processes of interest, their NLO results, which we gave in eqn. (8.41), are sufficient for N$^3$LL precision. The NNLO $I_{q/q}$ kernel we already provided in [40], the other kernels are obtained and presented for the first time.

In the next section, we will discuss the structure of our results and various strong checks, we performed to confirm their correctness.

## 8.5 Checks

In the last section, we presented the main results of our work, the NNLO matching kernels and anomaly coefficients. Even though written in an elegant form, most of the expressions are quite lengthy. Hence, a fast gaze at them will not confirm their correctness. In the calculation leading to them, we have already done many checks, including among others independent calculations, numerical checks for several integration steps, confirmed limits of $z$ dependent integrals and consistency checks between the results from the collinear and anti-collinear region. Still there are many more checks we can perform for our results, which we will discuss in this section. We will address a list of such aspects in the following, starting on a technical level, moving to the framework and consistency level and finally compare to results in the literature.

### 8.5.1 Mathematical Structure

A first point we can study for our results is their mathematical structure.

The functions of $z$ we encounter are $\delta$-functions, associated plus distributions, harmonic polylogarithms up to weight 3 and simple rational functions. These are exactly the class of functions expected for our NNLO calculation.

Despite the poles parametrized by the $\delta$-functions, the divergences of each function for $z \to 1$ are at most logarithmic. Moreover, the functions multiplied by $z$ contain at most logarithmic divergences for $z \to 0$.

Also the other variables appear in the form, as expected. We usually find $\zeta$-values only up to $\zeta_3$. The only place where $\zeta_4$ is found is in the diagonal splittings for the terms with a $\delta$ function and the color structure $C_a^2$ for the gluon and $C_f^2$ as well as $C_aC_f$ for the quark, respectively, which natively generate the highest poles and $\zeta$ functions.

The dependence on the transverse scale appeared solely in quadratic form and is contained in $L_\perp$. This is also the only place where mass scales enter. In particular, all dependence on the hard scale as well as $\bar{n}\cdot p$ and $n\cdot\bar{p}$ canceled out.

Moreover, for each kernel, only the expected combinations of two color factors appear.

Hence, all these attributes are as expected and the results have a consistent structure.

### 8.5.2 Symmetry

In section 4.7, we discussed the equality of the matching kernels of certain combinations of partons. In the last section, we therefore only listed the results of the independent functions. However, we also extracted results for other combinations of partons and found the expected agreement. This agreement contributes as a check to our results.

While the nTPDFs differed between the collinear and anti-collinear region, for the TPDFs, this distinction did not sustain, but the symmetry between the two regions was consistently recovered.

Another relation our results fulfill is the Casimir scaling of the anomaly coefficients, i.e. we found up to NNLO $F_{gg}/C_a = F_{q\bar{q}}/C_f$.

### 8.5.3 Collinear Anomaly and Renormalization

Another very important check of algebra, consistency and the framework as a whole is the cancellation of poles in the analytic regulator $\alpha$ and of the associated scale $v$. Both of them where present for the nTPDFs. However, as expected from general arguments and explained for the explicit results in section 8.1, both of them cancel in the combination of the collinear and corresponding anti-collinear functions. The generated dependence on $L_Q$ is entirely controlled by two functions $F_{i\bar{\imath}}$ consistently between all cases. These functions only depend on $\alpha_s$ and $L_\perp$. Also the $I$ kernels extracted from there consistently only depend on the single ratio of mass scales contained in $L_\perp$. Both the $F$ and $I$ functions are completely independent of the regulator $\alpha$ and the scale $v$. This shows that the analytic regularization could consistently be applied and that the physically relevant combinations (3.16,3.46) do not depend on it.

In addition to the poles in $\alpha$, also poles in $\epsilon$ have been present for the bare nTPDFs. The proposed operator renormalization and renormalization of the coupling consistently removed all these poles. It should be noted that while we extracted the renormalization factors explicitly from our bare results, they are also implied by their RGE equations and the $\overline{\text{MS}}$ condition in terms of the known anomalous dimensions and the QCD $\beta$ function, as pointed out in the end of section 3.1.7. Since both expressions agree, this is on the one hand another powerful check of our results and on the other hand allows us to rederive the anomalous dimensions and DGLAP kernels up to $\alpha_s^2$.

### 8.5.4 RGE Equations

Deeply connected to the renormalization is the RG evolution of these functions. As discussed in section 3, the consistency of the theory implies the RGE equations as

$$\frac{d}{d\log\mu} F_{i\bar{\imath}}(x_T^2,\mu) = 2\,\Gamma^i(\alpha_s) \tag{8.51}$$

and

$$\frac{d}{d\log\mu} I_{i/j}(z,x_T^2,\mu) = \Big[\Gamma^i(\alpha_s)L_\perp - 2\gamma^i(\alpha_s)\Big] I_{i/j}(z) - \sum_k 2 I_{i/k}(z)\otimes P_{k/j}(z)\,, \tag{8.52}$$

where $\Gamma^i$ is the cusp anomalous dimension in the representation corresponding to the parton $i$, $\gamma^i$ is the quark or gluon anomalous dimension, respectively, and $P_{k/j}$ are the DGLAP splitting kernels. On the RHS we suppressed the arguments $x_T^2$ and $\alpha_s$. In our results, $I_{i/j}^{(n)}$ are coefficients of $[\alpha_s(\mu)/(4\pi)]^n$ given as function of $z$ and $L_\perp(\mu)$. Hence, from our results the LHS can be determined straightforwardly using

$$\frac{dL_\perp}{d\log\mu} = 2\,, \qquad \frac{d\alpha_s(\mu)}{d\log\mu} = \beta(\alpha_s), \tag{8.53}$$

and eqn. (C.4). For the first part on the RHS, we plug in the constants listed in section C.1. The last term on the RHS is a Mellin convolution, which can include distributions. To those, we gave several remarks in section B.5. We explicitly confirmed that our functions obey these RGE equations, which is yet another strong check of our results.

We want to stress that the finding of consistent NNLO results, which appropriately refactorize, give rise to the collinear anomaly, renormalize in the highly specified way and fulfilling the expected RGE equations, is not only a very strong evidence of the correctness of our results but also of the framework as a whole.

### 8.5.5 Comparison to Literature

Finally, we can also compare to results existing in the literature. We already mentioned, that we extracted the renormalized PDF up to $\alpha_s^2$ as the $z$-dependent renormalization kernels of the $I$ functions. From them we obtained the DGLAP splitting kernels up to $\alpha_s^2$. They agree with the well know results of [41, 42].

Our NLO results for the $I$ and $I'$ functions as well as the NLO and NNLO $F$ functions agree with the results given earlier in [5, 6].

The NNLO $I$ functions are connected to the $\mathcal{H}^{(2)}(z)$ functions presented in [8, 9]. These process dependent coefficients correspond to the first factor in eqn. (3.34). We obtain the corresponding expressions from our perturbative functions $C_{q\bar{q}\leftarrow ij}$ and $2\pi C_{gg\leftarrow ij}$ by removing all logarithms of mass ratios and the integral over $x_\perp$. The former is obtained by evaluating the hard function at the hard scale $M$ or $m_H$ respectively and the $I$ functions at the scale $\mu_x = 2e^{-\gamma_E}/x_T$. This is, we have

$$\mathcal{H}^{DY}_{q\bar{q}\leftarrow jk}(z,\alpha_s) = \left|C_V(-M^2-i\epsilon, M)\right|^2 I_{q/j}(z, x_T^2, \mu_x) \otimes I_{\bar{q}/k}(z, x_T^2, \mu_x)\,, \tag{8.54}$$

$$\begin{aligned}\mathcal{H}^{H}_{gg\leftarrow jk}\big(z,\alpha_s,\log\tfrac{m_t^2}{m_h^2}\big) =& C_t^2(m_t^2, m_h)\left|C_S(-m_h^2-i\epsilon, m_h)\right|^2 \\ &\times \left[I_{g/j}(z, x_T^2, \mu_x) \otimes I_{g/k}(z, x_T^2, \mu_x) + I'_{g/j}(z, x_T^2, \mu_x) \otimes I'_{g/k}(z, x_T^2, \mu_x)\right].\end{aligned} \tag{8.55}$$

The scale choices for the hard functions are such that the scale logarithms in section C.4 result into $-i\pi$ for $C_V$ as well as $C_S$ and into $\log\frac{m_t^2}{m_h^2}$ for $C_t$. For the (anti)-collinear functions they are such that $L_\perp = 0$.

Note, as pointed out after eqn. (3.61), the LO expressions of $I'$ vanish and therefore the NLO expressions of $I'$ suffice to obtain the NNLO results of $\mathcal{H}^H_{gg\leftarrow jk}$.

For the Mellin convolution of distributions, appearing above, we again make use of the relations in section B.5. Setting $T_f = \frac{1}{2}$ and converting to the basis of polylogarithms used in [8, 9], we find exact agreement.

Since our results and the results in these references have been derived in a very different calculation and even a different framework, this agreement is a very strong check for both frameworks and calculations. A main advantage of our calculation is that it is performed directly from first principles, as it uses the gauge invariant operator definition obtained in the factorization theorem. Once we also supply the NNLO expressions for the $I'$ functions, we moreover will have extracted the general set of splitting kernels sufficing also for the most general spin states.

The long list of checks of very diverse nature, our results passed, compile to a strong confirmation of our results and the consistency of the framework we are using.

# 9 Conclusions

In the LHC era, high precision calculations for physics at hadron colliders are highly demanded. In general a corresponding fixed order calculation is insufficient, as it is spoiled by the presence of large logarithms of scale ratios. These logarithms have to be consistently resummed to all orders. Our interest is the resummation of logarithms of transverse momentum. While parton showers reach LL, but usually not full NLL precision, several classes of observables have been determined to this and a few even to higher logarithmic order by other, often (semi)-analytical, methods. The number of processes, in which $\mathrm{N}^3\mathrm{LL}$ precision is reached, is small. In terms of the NNLO $I$ functions, we determine functions, required for this precision in the wide range of processes, in which a color neutral final state of high invariant mass and small transverse momentum is produced.

We presented the general framework of effective field theories which allows for a consistent treatment of physics at different energy scales and eventually the resummation of their ratios. We then focused on the kinematically and calculationally involved case of the production of heavy states with small transverse momenta, for which Sudakov double logarithms of the ratio of these two scales appear. We discussed the in this case appropriate effective field theory SCET, and applied it to rederive factorization theorems for Drell-Yan and Higgs production along the lines of [5, 6]. We pointed out that similar theorems can straightforwardly be obtained for the production of any color neutral final state with high invariant mass and small transverse momentum.

The factorization theorems directly lead us among others to the operator definition of the nTPDFs. These objects have been the main focus of our work. They are generalizations of collinear PDFs, required to consistently describe the physics of the incoming hadrons up to the relevant energy scale and especially their effect on the transverse momentum of the considered final state, which is caused by recoiling against the initial state radiation.

While a factorization theorem and TPDFs have been known for a long time [7], the coherent extension of the theorem to all perturbative orders and the explicit determination of the TPDFs to high perturbative order have been a long standing problem.

Obtaining an all order factorization theorem and explicit operator definitions of the objects of our interest are great virtues of the SCET approach [5, 6] and are crucial for the determination of the NNLO corrections to the TPDFs, which is the main objective of this thesis.

This calculation suffered from rapidity singularities and needed additional regularization beyond dimensional one. We used the analytic regulator suggested in [10], to consistently perform the calculation of the parton-to-parton nTPDFs up to NNLO. The presence of the additional regulator and the very special integral kernels lead to unusual integrals, for which many standard methods could not be applied in a helpful way. We discussed the calculation of all integrals to the required order in the regulators in detail and from their results extracted the NNLO nTPDFs.

Within the framework, applied by us, the dependence on both regulators could consistently be removed. In one of the corresponding steps, a collinear anomaly [5] revealed a dependence of the product of two nTPDFs on the large invariant mass of the final state. From this product, we identified the anomaly coefficients $F$ and the refactorized TPDFs. Relating the latter to the collinear PDFs, we moreover extracted the matching kernels $I$ and $I'$. In this way, we determined both $F$ and $I$ up to NNLO in perturbation theory for all possible parton configurations. The $I'$ functions we provided to NLO and, in addition to that, discussed all ingredients necessary for their NNLO determination. All of these process independent functions are relevant for the high order resummation for a wide range of processes.

We also discussed a number of powerful checks, we applied to confirm the correctness of our results. As a byproduct of one of these checks, we confirmed the $\mathcal{H}^{(2)}$ coefficients presented in [8, 9].

The six independent NNLO $I$ functions are our main results. It is the first time, they have been determined. Their consistent extraction also explicitly demonstrates the consistency of the framework and the practical applicability of it and the analytic regulator at high perturbative order.

When we rederived the factorization theorems, we also discussed how the large logarithms can be resummed by solving the RGE equations. The functions extracted by us are, therefore, important ingredients for high-order resummation in the wide class of process, in which color neutral final states of high invariant mass and small transverse momentum are produced at hadron colliders. In addition to that, there are also indications for the applicability of our results in the $\mathrm{N^3LL}$ resummation of color charged final states, as in top-quark pair production [43]. Taking into account the currently known perturbative order of all expressions necessary to obtain the final resummed result, NNLL precision can be achieved. Among others, the NNLO $F$, the NLO $I'$ and $I$ function, are needed to obtain this precision. Our NNLO results of the $I$ functions even provide generic ingredients for $\mathrm{N^3LL}$ resummation. In most cases of practical interest, like Higgs production, many new physics scenarios, production of individual or multiple vector bosons and all other $q\bar{q}$-initiated processes, also the NLO $I'$ functions are already sufficient for this precision. Only for the gluon induced production of final states with a more complicated spin structure than the one of a Higgs boson, also the NNLO $I'$ functions are required to obtain $\mathrm{N^3LL}$ precision. We already presented all necessary ingredients for the extraction of these functions and will, presumably in close future, provide their NNLO results and by this another relevant ingredient for $\mathrm{N^3LL}$ resummation. As pointed out in section 3.1.8, some of the other ingredients for the $\mathrm{N^3LL}$ resummation are not yet known to sufficient perturbative order. This includes the $F$ functions, which are needed up to $\mathrm{N^3LO}$.

Beyond the $q_T$ resummation, the TPDFs and their matching kernels are also relevant for other aspects of QCD. For example, they are important to describe spin or azimuthal related observables as well as to understand the spin structure of the proton and other hadrons.

Obtained directly from the operator definitions in the factorization theorem, our NNLO results are the first achieved directly from first principles. Having contributed important results towards $N^3LL$ resummation and other aspects in QCD, we are looking forward to applications of our results in high precision physics.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# A Partial Fraction Decomposition

In this section we discuss partial fraction decomposition (PF) which is a very generic method to simplify expressions by splitting apart factors depending on the same variable. In its basic form it is rewriting

$$1 = (1-y) + y \;\Rightarrow\; \frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y} \quad \text{or} \tag{A.1}$$

$$\frac{1-y}{y} = \frac{1}{y} - 1\,, \; \frac{y}{1-y} = \frac{1}{1-y} - 1\,. \tag{A.2}$$

If two denominators $y$ and $(1-y)$ appear with integer powers, we can iteratively apply the equations above and therefore write this term as a sum of terms of which each contains only one of the two denominators.
More generally for two denominators $D_i$ and $D_j$ one can write,

$$1 = b_{ij}D_j + c_{ij}D_i \Rightarrow \tag{A.3}$$

$$D_i^{-1}D_j^{-1} = b_{ij}D_i^{-1} + c_{ij}D_j^{-1}\,, \tag{A.4}$$

$$D_i^{-1}D_j = \frac{1}{b_{ij}}D_i^{-1} - \frac{c_{ij}}{b_{ij}}\,, \tag{A.5}$$

$$D_iD_j^{-1} = \frac{1}{b_{ij}}D_j^{-1} - \frac{b_{ij}}{c_{ij}}\,, \tag{A.6}$$

where the values of $b_{ij}$ and $c_{ij}$ obviously depend on the denominators. Because those values can be complicated functions of the variables of the denominators, the relations are only helpful for appropriate combinations of denominators for which $b_{ij}$ and $c_{ij}$ are notably simpler than the denominators themselves. Writing $I(a_i, a_j) = D_i^{-a_i}D_j^{-a_j}$, the relations above are equivalent to

$$I(a_i, a_j) = b_{ij}I(a_i, a_j - 1) + c_{ij}I(a_i - 1, a_j)\,, \tag{A.7}$$

$$I(a_i, a_j) = \frac{1}{b_{ij}}I(a_i, a_j + 1) - \frac{c_{ij}}{bi}I(a_i - 1, a_j + 1)\,, \tag{A.8}$$

$$I(a_i, a_j) = \frac{1}{c_{ij}}I(a_i + 1, a_j) - \frac{b_{ij}}{c_{ij}}I(a_i + 1, a_j - 1)\,, \tag{A.9}$$

Each equation is valid for any value of $a_{i,j}$. They can be applied recursively to each individual integral in a linear combination of integrals aiming to reduce the number or complexity of individual integrals. The optimal way in which this is done depends on the exact structure of the integral and is to some extend arbitrary. Our **default tactic** is to apply

- eqn. (A.7) if both $a_i$ and $a_j$ are positive;
- eqn. (A.8) if $a_j$ is negative and $a_i$ positive;
- or eqn. (A.9) in the reversed case.

This will ensure that at least one index is 0. Moreover, one can use one of the last two equations to ensure that one of the two indices never is negative.

In our discussion here we assumed that $a_i$ and $a_j$ are integers. In many applications they are actually shifted from the integers by regulators. Obviously the relations above also hold for such cases, however, statements like positive/negative/0 should be understood as up to regulators.

# B Special Functions

In the calculation of multi-loop corrections, a number of special functions appear. We will here discuss some properties of a choice of special functions most relevant for us.

## B.1 Gamma Functions

A function we need at many places in our calculation is the Gamma function, which can be represented by the following integral

$$\Gamma(z) = \int_0^\infty \frac{\mathrm{d}t}{t} t^z e^{-t} \,. \tag{B.1}$$

It is analytic for all $z \in \mathbb{C}$ with the exception of non-positive integer values, where it has poles. For positive integer numbers it is simply the factorial $n! = \Gamma(n+1)$.

Some relevant equations, we use in our calculation are:

$$z\Gamma(z) = \Gamma(z+1)\,, \tag{B.2}$$

$$\Gamma(z+\tfrac{1}{2}) = 2^{1-2z}\sqrt{\pi}\frac{\Gamma(2z)}{\Gamma(z)}\,, \tag{B.3}$$

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}\,. \tag{B.4}$$

The first equation helps to simplify expressions or to extract poles from the Γ-function. Similarly the second and third eq. allows us to eventually simplify expressions or bring them to a common form.

## B.2 Harmonic Polylogarithms

In this section we repeat the definition of harmonic polylogarithms (HPLs). We follow [38, 44], where additional information can be found. The harmonic polylogarithms $H_{\vec{a}_n}(z)$ of weight $n$ are functions of $z$ and labeled by a $n$-dimensional vector $\vec{a}_n = (a_1, \ldots, a_n)$ with $a_i \in \{-1, 0, 1\}$. At weight $n = 1$ they are given by

$$\begin{aligned} H_{-1}(z) &= \int_0^z \mathrm{d}x\, \frac{1}{1+x} = \log(1+z)\,, \\ H_0(z) &= \log z\,, \end{aligned}$$

$$H_1(z) = \int_0^z \mathrm{d}x \, \frac{1}{1-x} = -\log(1-z)\,. \tag{B.5}$$

At weight $n$ we have

$$H_{\vec{0}_n}(z) = \frac{1}{n!}\log^n z, \tag{B.6}$$

while all other HPLs can be written as iterated integrals with the three functions $f_a(x)$, given by

$$\begin{aligned} f_{-1}(x) &= \frac{1}{1+x}\,, \\ f_0(x) &= \frac{1}{x}\,, \\ f_1(x) &= \frac{1}{1-x}\,, \end{aligned} \tag{B.7}$$

and HPLs of a weight decreased by 1. More explicitly for $\vec{a}_n \neq \vec{0}_n$,

$$H_{\vec{a}_n}(z) = \int_0^z \mathrm{d}x \, f_{a_n}(x) H_{\vec{a}_{n-1}}(x)\,, \tag{B.8}$$

where $a_n$ is the leftmost component of $\vec{a}_n$, which has been removed in the vector $\vec{a}_{n-1}$. As a consequence of this definition, the derivatives take the form

$$\frac{d}{dz} H_{\vec{a}_n}(z) = f_{a_n}(z) H_{\vec{a}_{n-1}}(z)\,. \tag{B.9}$$

This class of functions embeds the Nielsen polylogarithms and the usual polylogarithms. The HPLs fulfill many nice properties and relations. Some of them relate HPLs of different arguments. For brevity, we will not provide repeat these relations. They can be found in the references above. Let us repeat another relation, however. Given two HPLs of weight $n$ and $m$ labeled by the vectors $a_n$ and $b_m$, their product can be rewritten as

$$H_{\vec{a}_n}(z) H_{\vec{b}_m}(z) = \sum_{\vec{c}_k \in \vec{a}_n \uplus \vec{b}_m} H_{\vec{c}_k}(z)\,, \tag{B.10}$$

where the shuffle $\vec{a}_n \uplus \vec{b}_m$ is the set of all combinations of all elements of $\vec{a}_n$ and $\vec{b}_m$, which leaves the internal order of each of them unchanged. We therefore can write the product of the two HPLs as sum of single HPLs with weight $k = n + m$.

When we provide our perturbative results, we will suppress the argument $z$ for brevity. In our results, we also encounter Zeta-values $\zeta_n$. They are given by the limits of some HPLs for $z \to 1$. More explicitly for $n > 1$

$$\zeta_n = H_{0_{n-1},1}(1)\,. \tag{B.11}$$

These constants are more commonly represented by the series

$$\zeta_n = \sum_{k=1}^{\infty} k^{-n} \,. \tag{B.12}$$

The limits for $z \to 1$ of HPLs with different label vectors, are described by colored Multiple Zeta values, but are not relevant in our calculation.

## B.3 Hypergeometric Functions

In this section we will discuss properties of Hypergeometric functions. We will mostly focus on the Gauss Hypergeometric function ${}_2F_1$, but also consider some aspects of the more general set of functions ${}_pF_q$. Our references, in which also many additional aspects can be found, are [45,46].

### B.3.1 Definition and Analytic Properties

The generalized Hypergeometric function ${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)$ is defined as the solution $w(z)$ of the differential equation

$$z\prod_{n=1}^{p}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+a_n\right)w(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\prod_{n=1}^{q}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+b_n-1\right)w(z). \tag{B.13}$$

With the Pochhammer-symbol $(a_n)_k = \frac{\Gamma(a_n+k)}{\Gamma(a_n)}$ we can represent this function by the series

$${}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};z\right) = {}_pF_q\left(a_1,\dots,a_p;b_1,\dots,b_q;z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!} \tag{B.14}$$

if non of the $b_i$ is a non-positive integer. From this representation it is evident that we can shuffle the parameters in the two sets $\{a_1,...,a_p\}$ and $\{b_1,...,b_q\}$ at will and that we can cancel two parameters $a_i$ and $b_j$ if they are equal (and not non-positive integers), which then reduces a ${}_pF_q$ to a ${}_{p-1}F_{q-1}$. This means

$${}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ a_1,b_2\dots,b_q\end{matrix};z\right) = {}_{p-1}F_{q-1}\left(\begin{matrix}a_2,\dots,a_p\\ b_2\dots,b_q\end{matrix};z\right) \tag{B.15}$$

if without loss of generality the equal parameters are $a_1 = b_1 \notin -\mathbb{N}_0$.

If any $a_i$ but no $b_j$ is a non-positive integer, then only the first $-a_i$ terms of the series in eqn. (B.14) are different from 0, as is implied by the definition via the Pochhammer symbols and the equation $\Gamma(z+1) = z\Gamma(z)$. Then the corresponding Hypergeometric function is a polynomial of degree $-a_i$ in $z$ with infinite radius of convergence. If the largest non positive integer is $a_1 = -n$ and $b_j \notin -\mathbb{N}_0$, we thus obtain the polynomial

$$ {}_pF_q\left(\begin{matrix}-n, a_2, \ldots, a_p \\ b_1, \ldots, b_q\end{matrix}; z\right) = \sum_{k=0}^{n} \frac{(-1)^k n!\,(a_2)_k \cdots (a_p)_k}{(n-k)!\,(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!} \,. \tag{B.16} $$

For the generic case without non-positive integer valued parameters, the analytic properties depend on the number and values of the parameters in the following way.

If $p \leq q$, the series converges for all finite values of $z$. If $p > q+1$, the series in general diverges for any non-zero value of $z$. For us, the most interesting case is $p = q+1$. Then the function's radius of convergence is 1. Outside this circle, the function can be defined by analytic continuation w.r.t. $z$. There its value is not uniquely defined, as the function has a branch cut. For $|z| = 1$ the series is absolutely convergent for $\Re(\gamma) > 0$, convergent except at $z = 1$ if $-1 < \Re(\gamma) \leq 0$, and divergent if $\Re(\gamma) \leq -1$, where

$$ \gamma = (b_1 + \ldots + b_q) - (a_1 + \ldots + a_{q+1}) \,. \tag{B.17} $$

### Analytic Continuation and other Relations

Instead of providing general equations for analytical continuation, we will only provide the equation for ${}_2F_1$, which we actually have used. In cases where $|x| > 1$, we can use the following identity to receive the new argument $x^{-1}$, which then lies in the radius of convergence:

$$ \begin{aligned} {}_2F_1(a, b; c; x) =& \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-x)^{-a}\, {}_2F_1(a, 1+a-c; 1+a-b; x^{-1}) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-x)^{-b}\, {}_2F_1(b, 1+b-c; 1-a+b; x^{-1}) \,, \end{aligned} \tag{B.18} $$

which requires $|\arg(-x)| < \pi$. In the book also three more mappings to ${}_2F_1$ with arguments $1-z$, $(1-z)^{-1}$ and $1-z^{-1}$ are given. The identities for the last two cases obviously follow from the first two.

For cases where $\gamma = c - a - b < 0$, we can use the so called Euler-transformation,

$$ {}_2F_1(a, b; c; x) = (1-x)^{c-a-b}\, {}_2F_1(c-a, c-b; c; x) \,, \tag{B.19} $$

to extract the divergent part at $x = 1$. This leads to $\gamma_{\text{new}} = -\gamma_{\text{old}} > 0$, hence the ${}_2F_1$ function on the RHS converges at $x = 1$.

For a general analytic ${}_pF_q$ the $n$-th derivative is given by

$$ \frac{d^n}{dz^n}\, {}_pF_q\left(\begin{matrix}a_1, \ldots, a_p \\ b_1, \ldots, b_q\end{matrix}; z\right) = \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n}\, {}_pF_q\left(\begin{matrix}a_1+n, \ldots, a_p+n \\ b_1+n, \ldots, b_q+n\end{matrix}; z\right) \,, \tag{B.20} $$

which is directly implied by the series representation eqn. (B.14).

Of course, there are many more relations. A big subgroup of relations are those among contiguous functions, where two functions ${}_pF_q$ are called contiguous if both have the same numbers $p$ and $q$ of parameters and the individual parameters differ by integers.

However, we will not recall any of those equations here. The only other relation we recall is the quartic transformation

$$ {}_2F_1(a,b;2b,4z(1+z)^{-2}) = (1+z)^{2a}{}_2F_1(a,a+\tfrac{1}{2}-b;b+\tfrac{1}{2},z^2) \ , \tag{B.21} $$

which is valid for appropriate $a$, $b$ and if $|z|<1$.

### B.3.2 Integral Representation

Instead of by eqn. (B.14), we can represent the Hypergeometric functions by integrals. One option to do this is in a recursive way via

$$ {}_{p+1}F_{q+1}\left(\begin{matrix} a_0,\ldots,a_p \\ b_0,\ldots,b_q \end{matrix};z\right) = \frac{\Gamma(b_0)}{\Gamma(a_0)\Gamma(b_0-a_0)}\int_0^1 \mathrm{d}t\, t^{a_0-1}(1-t)^{b_0-a_0-1}{}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix};zt\right) \ , \tag{B.22} $$

for $\Re(b_0)>\Re(a_0)>0$. This holds for $p\leq q$ or if in addition $|\arg(1-z)|<\pi$, for $p=q+1$.

Functions ${}_pF_q$ with low number of arguments are for example

$$ {}_1F_0\,(a;\,;z) = (1-z)^{-a} \ , \tag{B.23} $$
$$ {}_0F_0\,(\,;\,;z) = e^z \ . \tag{B.24} $$

In our calculation we often encounter the following integral, which is by above equations proportional to ${}_2F_1$,

$$ \int_0^1 \mathrm{d}t\, t^a(1-t)^b(1-z_-(1-t))^c = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}{}_2F_1(-c,b+1;a+b+2;z_-) \ , \tag{B.25} $$

for $z_-\notin[1,\infty)$ and $\Re(a),\Re(b)>-1$, which is in agreement with the range required for (B.22). For $z_-=1$ we need $a+c>-1$. In the limit $c=0$ or $z_-=0$, the left hand side reduces to a $\beta$-function, because the third denominator is not present anymore. Also on the right-hand side ${}_2F_1(\ldots,0)$ reduces to 1, in agreement with the integral representation of the $\beta$-function

$$ \beta(x,y) = \int_0^1 \mathrm{d}t\, t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \ , \tag{B.26} $$

for $\Re(x),\Re(y)>0$.

## B.4 Distributions

In our calculation we often want to solve integrals of products of two function of which one has a single pole in the integration range. Be $f$ one of those functions with a single at $a$ in the integration range $[b,c]$ and finite elsewhere. Be $g$ the other function on $[b,c]$ which is finite and smooth in $a$. By adding $0 = g(a) - g(a)$, we then can write

$$\int_b^c \mathrm{d}x\, f(x)g(x) = g(a)\int_b^c \mathrm{d}x\, f(x) + \int_b^c \mathrm{d}x\, f(x)\big[g(x) - g(a)\big]\,. \tag{B.27}$$

The second integral then does not contain the pole at $a$ anymore and in many cases the first integral can be solved analytically. Let us call the corresponding result $I(f,b,c) = \int_b^c \mathrm{d}x\, f(x)$. Then a convenient way to implicitly write the last equation is in the form

$$f(x) = I(f,b,c)\delta(x-a) + \big[f(x)\big]_{+,a}^{[b,c]} \tag{B.28}$$

with the delta distribution $\delta$ and the plus distribution $[\ldots]_{+,a}^{[b,c]}$. Both are still understood as functions in an integral over $x$ from $b$ to $c$. For the two functions $f(x), g(x)$ above we have

$$\int_b^c \mathrm{d}x\, \delta(x-a)g(x) = g(a)\,, \quad \text{and} \tag{B.29}$$

$$\int_b^c \mathrm{d}x\, \big[f(x)\big]_{+,a}^{[b,c]} g(x) = \int_b^c \mathrm{d}x\, f(x)\big[g(x) - g(a)\big]\,. \tag{B.30}$$

Be aware that the notation in eqn. (B.28) is not completely rigorous. It should be always understood as appearing below the corresponding integral with range $[a,b]$. Nevertheless, in some cases we will consider the RHS of eqn. (B.28) below an integral of different range as underlying the splitting in the two separate terms, then one has to carefully correct for this change. Considering the difference eqn. (B.28) for two different lower boundaries $b$ and $b'$, we can write

$$\big[f(x)\big]_{+,a}^{[b,c]} = \big[f(x)\big]_{+,a}^{[b',c]} - I(f,b,b')\delta(x-a) \tag{B.31}$$

This will be relevant, when we consider the convolution of terms including such distributions.

In our calculation we often have integrals from 0 to 1 containing the factor $(x-a)^{-1+r}$ with $r$ infinitesimally small and $a \in [0,1]$. Following eqn. (B.28), we then usually write

$$(x-a)^{-1+r} = \frac{1}{r}\delta(x-a) + \big[(x-a)^{-1+r}\big]_+\,, \tag{B.32}$$

where both factors have support in $[0,1]$ and the latter term will often be expanded in $r$ to some finite order. For sake of simplicity we used a flattened and more common notation.

Above and throughout this work this is

$$\left[(x-a)^{-1+r}\right]_{+} \equiv \left[(x-a)^{-1+r}\right]_{+,a}^{[0,1]} . \tag{B.33}$$

Obviously, we did not provide a complete, mathematical rigorous discussion of distributions here. Such discussions can be found in standard mathematics books.

## B.5 Convolution Integrals

For the Mellin convolution of two functions, we use the short hand notation

$$\begin{aligned} f(z) \otimes g(z) &\equiv \int_z^1 \frac{\mathrm{d}x}{x} f(x) g(z/x) \\ &= \int_z^1 \frac{\mathrm{d}x'}{x'} g(x') f(z/x') = g(z) \otimes f(z) . \end{aligned} \tag{B.34}$$

For smooth functions for which the product has no poles in $[z,1]$ this is a straight forward definition. However, if we allow $f$ and/or $g$ to be distributions, some comments are in place. The issue is that in eqns. as (B.32) implicitly the range of integration is assumed to be $[0,1]$. If the individual terms appear in integrals with different range, one has to consistently adjust the result for the different range. We will do this for the terms with the plus-prescription.

For convolutions including the Dirac $\delta$, we then simply have

$$\delta(z-a) \otimes g(z) = \begin{cases} g(z/a)/a , & \text{if } a \in [z,1] \\ 0 , & \text{if } a \notin [z,1] . \end{cases} \tag{B.35}$$

This also holds if $g$ is a distribution itself.

For $[f(x)]_{+,a}^{[0,1]}$ with support $[0,1]$ and the pole at $x = a \neq 0$ and a smooth function $g(x)$ we, however, have

$$[f(z)]_{+,a}^{[0,1]} \otimes g(z) = \int_z^1 \mathrm{d}x\, f(x) \left[\tfrac{1}{x} g\left(\tfrac{z}{x}\right) - \tfrac{1}{a} g(a)\right] - \tfrac{1}{a} g(a) \int_0^z dx f(x) , \tag{B.36}$$

where the latter term is the implication of eqn. (B.31) with $f(x)$ replaced by $g(x)/x$, $c = 1$, $b = 0$ and $b' = z$. If both functions are plus distributions, we use the implication of eqn. (B.31) for each of them iteratively. For the case explicitly relevant in our calculation, we then find

$$\left[\frac{1}{1-z}\right]_{+,1}^{[0,1]} \otimes \left[\frac{1}{1-z}\right]_{+,1}^{[0,1]} = 2\left[\frac{\log(1-z)}{1-z}\right]_{+} - \frac{\log z}{1-z} - \zeta_2\, \delta(1-z) . \tag{B.37}$$

# C Constants

In this appendix we collect several known expressions which we use to perform various checks for our result. This includes anomalous dimensions, the QCD $\beta$-function, splitting functions and Wilson coefficients. Throughout this section we suppress the argument $\mu$ of the renormalized strong coupling $\alpha_s$.

## C.1 Anomalous Dimensions

We define the perturbative expansion of the quark and gluon anomalous dimensions $\gamma^i$ as

$$\gamma^i(\alpha_s) = \frac{\alpha_s}{4\pi}\,\gamma_0^i + \left(\frac{\alpha_s}{4\pi}\right)^2 \gamma_1^i + \mathcal{O}(\alpha_s^3)\,, \tag{C.1}$$

and analogously for the cusp anomalous dimensions $\Gamma^i$ in the fundamental and adjoint representation. The coefficients up to the second order are given by

$$\begin{aligned}
&\frac{1}{C_f}\Gamma_0^q = \frac{1}{C_a}\Gamma_0^g = 4\,,\\
&\frac{1}{C_f}\Gamma_1^q = \frac{1}{C_a}\Gamma_1^g = \left(\frac{268}{9} - \frac{4\pi^2}{3}\right) C_a - \frac{80}{9} T_f N_f\,,\\
&\gamma_0^q = -3C_f\,,\\
&\gamma_1^q = C_f^2\left(-\frac{3}{2} + 2\pi^2 - 24\zeta_3\right) + C_f C_a\left(-\frac{961}{54} - \frac{11\pi^2}{6} + 26\zeta_3\right) + C_f T_f N_f\left(\frac{130}{27} + \frac{2\pi^2}{3}\right),\\
&\gamma_0^g = -\frac{11}{3}C_a + \frac{4}{3}T_f N_f\,,\\
&\gamma_1^g = C_a^2\left(-\frac{692}{27} + \frac{11\pi^2}{18} + 2\zeta_3\right) + C_a T_f N_f\left(\frac{256}{27} - \frac{2\pi^2}{9}\right) + 4C_f T_f N_f\,.
\end{aligned} \tag{C.2}$$

## C.2 QCD $\beta$-Function

The perturbative expansion of the QCD $\beta$-function

$$\beta(\alpha_s) = \frac{d\alpha_s(\mu)}{d\log\mu}$$

is given by

$$\beta(\alpha_s) = -2\alpha_s \left[ \frac{\alpha_s}{4\pi}\,\beta_0 + \left(\frac{\alpha_s}{4\pi}\right)^2 \beta_1 + \mathcal{O}(\alpha_s^3) \right], \tag{C.3}$$

where

$$\begin{aligned} \beta_0 &= \frac{11}{3} C_a - \frac{4}{3} T_f N_f \,, \\ \beta_1 &= \frac{34}{3} C_a^2 - \frac{20}{3} C_a T_f N_f - 4 C_f T_f N_f \,. \end{aligned} \tag{C.4}$$

## C.3 Splitting Functions

The DGLAP splitting functions are

$$P_{ij}(z) = \frac{\alpha_s}{4\pi}\, P_{ij}^{(0)}(z) + \left(\frac{\alpha_s}{4\pi}\right)^2 P_{ij}^{(1)}(z) + \mathcal{O}(\alpha_s^3) \,, \tag{C.5}$$

with the first order coefficients

$$\begin{aligned} P_{qq}^{(0)}(z) &= 2C_f \tilde{p}_{qq}(z) + 3C_f \delta(1-z), \\ P_{gg}^{(0)}(z) &= 4C_a \tilde{p}_{gg}(z) + \left[\tfrac{11}{3} C_a - \tfrac{4}{3} T_f N_f\right] \delta(1-z) \,, \\ P_{qg}^{(0)}(z) &= 2T_f \tilde{p}_{qg}(z) \,, \\ P_{gq}^{(0)}(z) &= 2C_f \tilde{p}_{gq}(z) \,, \end{aligned} \tag{C.6}$$

with the functions

$$\begin{aligned} \tilde{p}_{qq}(z) &= \frac{1+z^2}{(1-z)_+} \,, \\ \tilde{p}_{gg}(z) &= \frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \,, \\ \tilde{p}_{qg}(z) &= z^2 + (1-z)^2 \,, \\ \tilde{p}_{gq}(z) &= \frac{1+(1-z)^2}{z} \,. \end{aligned} \tag{C.7}$$

The second order coefficients can be obtained from the results in [41, 42] (see also [28, 29]), and we do not repeat the long expressions here.

The splitting functions are related to the renormalized parton-to-parton PDFs by

$$\frac{d}{d\ln\mu}\,\phi_{i/j}(z,\mu) = \sum_k 2P_{ik}(z,\mu) \otimes \phi_{k/j}(z,\mu) \,. \tag{C.8}$$

This relation implies

$$
\begin{aligned}
\phi^{(1)}_{i/j}(z) &= -\frac{P^{(0)}_{ij}(z)}{\epsilon}\,, \\
\phi^{(2)}_{i/j}(z) &= \frac{1}{2\epsilon^2}\left[\sum_k P^{(0)}_{ik}(z)\otimes P^{(0)}_{kj}(z) + \beta_0\, P^{(0)}_{ij}(z)\right] - \frac{P^{(1)}_{ij}(z)}{2\epsilon}\,.
\end{aligned}
\tag{C.9}
$$

## C.4 Wilson Coefficients

The Wilson coefficient for Drell-Yan production is [20]

$$
C_V(-q^2,\mu) = 1 + C_f\frac{\alpha_s}{4\pi}\left(-L^2+3L-8+\zeta_2\right) + C_f\left(\frac{\alpha_s}{4\pi}\right)^2\left[C_f H_F + C_a H_A + T_f N_f H_f\right], \tag{C.10}
$$

where $L = \log\frac{-q^2}{\mu^2}$ and

$$
\begin{aligned}
H_F =& \frac{L^4}{2} - 3L^3 + \left(\frac{25}{2}-\frac{\pi^2}{6}\right)L^2 + \left(-\frac{45}{2}-\frac{3\pi^2}{2}+24\zeta_3\right)L + \frac{255}{8} + \frac{7\pi^2}{2} - \frac{83\pi^4}{360} - 30\zeta_3\,, \\
H_A =& \frac{11}{9}L^3 + \left(-\frac{233}{18}+\frac{\pi^2}{3}\right)L^2 + \left(\frac{2545}{54}+\frac{11\pi^2}{9}-26\zeta_3\right)L \\
& - \frac{51157}{648} - \frac{337\pi^2}{108} + \frac{11\pi^4}{45} + \frac{313}{9}\zeta_3\,, \\
H_f =& -\frac{4}{9}L^3 + \frac{38}{9}L^2 + \left(-\frac{418}{27}-\frac{4\pi^2}{9}\right)L + \frac{4085}{162} + \frac{23\pi^2}{27} + \frac{4}{9}\zeta_3\,.
\end{aligned}
\tag{C.11}
$$

For Higgs production the relevant Wilson coefficients $C_t$ and $C_H$ are both listed in [34] up to $\alpha_s^2$. We only give the terms up to $\alpha_s^1$ here, which are

$$
C_t(m_t^2,\mu^2) = 1 + \frac{\alpha_s}{4\pi}(5C_a - 3C_f)\,, \tag{C.12}
$$

$$
C_S(-m_h^2 - i\epsilon,\mu^2) = 1 + \frac{\alpha_s}{4\pi}C_a\Big(-\log^2\frac{-m_h^2-i\epsilon}{\mu^2} + \zeta_2\Big)\,. \tag{C.13}
$$

# D List of double real integrals

In this section we list the double real integrals which we needed in our calculation, except those whose result can be obtained easily in terms of Hypergeometric functions. The latter include all integrals with $a_2 \leq 0$, as then occurrences of $D_2$ can be expanded and the integral factorizes. It also includes integrals with $a_8, a_9 = 0$ if $\alpha$ can be dropped and either $a_7 = 0$ or $a_1, a_3 \leq 0$. In both cases we can use eq. (7.40, 7.41) and perform the remaining $y$ integral easily. In the latter case we express the ${}_2F_1$ with negative index in terms of the corresponding polynomial.
Of the remaining integrals we will first list the integrals whose expansion is needed up to $\alpha^1$, then we will list the integrals which contain a pole in $\alpha$ themselves. After that we will list the integrals with $a_2, a_1 + a_3, a_7 > 0$ but $a_8, a_9 = 0$ and finally the integrals with $a_8$ or $a_9 \neq 0$. For the integrals of the first group we choose as the second expansion parameter $\alpha/\epsilon$ and need to expand them up to $(\alpha/\epsilon)^1, \epsilon^2$. The integrals of all other groups need to be expanded to $\alpha^0, \epsilon^1$ only. For a couple of integrals an expansion to order $\alpha^0, \epsilon^0$ is sufficient. We write the results of the integrals in terms of $\zeta$-values and the harmonic polylogarithms $H_{\{m\}} \equiv H(\{m\}, z)$ introduced in [44].

The following integrals are non trivial, because their expansion is needed to $(\alpha/\epsilon)^1$. For all of them $a_7, a_8, a_9 = 0$ and we suppress those indices here. Moreover, we use $r = -\epsilon - \alpha(1-x)$, where $x = 1$ for the anti-collinear case and $x = -1$ for the collinear case. Then we receive

$$\begin{aligned}
I^{\mathrm{RR}}(\alpha, 1, \alpha, 0, r, 1+\epsilon) =& \frac{1}{\epsilon^2} - 2\epsilon\zeta_3 - 3\epsilon^2\zeta_4 - \frac{\alpha}{\epsilon}\left[\frac{1}{2\epsilon^2} + (1+x)\zeta_2 + \epsilon(4-x)\zeta_3 \right. \\
&\left. + \epsilon^2\frac{11+2x}{2}\zeta_4\right] + \mathcal{O}\big([\alpha/\epsilon]^2, \epsilon^3\big)\,, \\
I^{\mathrm{RR}}(1+\alpha, 1, \alpha, 0, r, \epsilon) =& \frac{1}{\epsilon^2} + 2\zeta_2 + 4\epsilon\zeta_3 + 11\epsilon^2\zeta_4 - \frac{\alpha}{\epsilon}\left[\frac{1}{\epsilon^2} + 2x\zeta_2 - 2\epsilon\zeta_3 - \epsilon^2\frac{27-17x}{2}\zeta_4\right] \\
&+ \mathcal{O}\big([\alpha/\epsilon]^2, \epsilon^3\big)\,, \\
I^{\mathrm{RR}}(\alpha, 1, 1+\alpha, 0, -1+r, 1+\epsilon) =& \frac{2}{\epsilon^2} - 2\zeta_2 - 6\epsilon\zeta_3 - 8\epsilon^2\zeta_4 - \frac{\alpha}{\epsilon}\left[\frac{3}{2\epsilon^2} + 2(1+x)\zeta_2 \right. \\
&\left. + \epsilon(11+2x)\zeta_3 + \epsilon^2\frac{56-x}{2}\zeta_4\right] + \mathcal{O}\big([\alpha/\epsilon]^2, \epsilon^3\big)\,,
\end{aligned}$$

$$I^{\mathrm{RR}}(\alpha,1,1+\alpha,0,r,\epsilon) = \frac{2x}{\alpha/\epsilon}\left[\frac{1}{\epsilon^2} - 2\epsilon\zeta_3 - 3\epsilon^2\zeta_4\right] - \frac{x}{\epsilon^2} - 2\zeta_2 + \epsilon(2-6x)\zeta_3 - \epsilon^2(2+9x)\zeta_4 + \frac{\alpha}{\epsilon}\left[\frac{x}{\epsilon^2} + \epsilon(2-4x)\zeta_3 - \epsilon^2\frac{15-5x}{2}\zeta_4\right] + \mathcal{O}\big([\alpha/\epsilon]^2, \epsilon^3\big)\,.$$

Obviously the last integral contains a pole in $\alpha$. The additional two integrals containing such poles are

$$I^{\mathrm{RR}}(\alpha,2,1+\alpha,0,-1-\epsilon,-1+\epsilon) = \frac{1}{\alpha/\epsilon}\left[\frac{2}{\epsilon^2}+\frac{4}{\epsilon}\right] - \frac{1}{\epsilon^2} + \frac{4}{\epsilon} - 2\zeta_2 + 9 + \mathcal{O}(\alpha/\epsilon,\epsilon)\,,$$
$$I^{\mathrm{RR}}(\alpha,1,2+\alpha,0,-1-\epsilon-2\alpha,\epsilon) = \frac{-1}{\alpha/\epsilon}\left[\frac{2}{\epsilon^2}+\frac{4}{\epsilon}\right] + \frac{1}{\epsilon^2} + \frac{2}{\epsilon} - 2\zeta_2 + 6 + \mathcal{O}(\alpha/\epsilon,\epsilon)\,.$$

For all integrals following below we can drop $\alpha$ and their results are understood as $\mathcal{O}(\alpha)$. We first list the remaining integrals with $a_2, a_1+a_3, a_7 > 0$ but $a_8, a_9 = 0$ multiplied by the factor $z_-$. They are

$$z_-\, I^{\mathrm{RR}}(1,1,0,0,-\epsilon,-1+\epsilon,1,0,0) = \frac{2H_0}{\epsilon} + 2\Big[H_{0,0} - H_{1,0} - \zeta_2\Big] + 2\epsilon\Big[H_{0,0,0} - 2H_{0,1,0} - H_{1,0,0} - 3\zeta_3\Big] + \mathcal{O}(\epsilon^2)\,,$$
$$z_-\, I^{\mathrm{RR}}(0,1,1,0,-1-\epsilon,\epsilon,1,0,0) = \frac{2H_0}{\epsilon} - 2\Big[H_{1,0} + \zeta_2\Big] - 2\epsilon\Big[H_{1,0,0} - \zeta_3\Big] + \mathcal{O}(\epsilon^2)\,,$$
$$z_-\, I^{\mathrm{RR}}(0,1,2,0,-2-\epsilon,\epsilon,1,0,0) = \frac{2H_0}{\epsilon} + 2\left[-H_{1,0} + H_0\left(1+\frac{1}{z_-}\right) - \zeta_2 + 1\right] + \mathcal{O}(\epsilon)\,.$$

These integrals and all following ones are at most logarithmically divergent at $z \to 1$. Because those integrals are never multiplied by $\delta([1-z])$ or $(f(z))_+$, the logarithmic divergences are unproblematic. The remaining integrals with $a_8$ or $a_9 \neq 0$ are

$$I^{\mathrm{RR}}(-1,2,-1,0,-\epsilon,\epsilon,0,1,0) = 0 + \mathcal{O}(\epsilon^2)\,,$$
$$I^{\mathrm{RR}}(-1,2,-1,0,-\epsilon,\epsilon,0,2,0) = 0 + \mathcal{O}(\epsilon^2)\,,$$
$$z_-\, I^{\mathrm{RR}}(-1,1,0,0,-\epsilon,1+\epsilon,0,1,0) = \frac{2H_0}{\epsilon} - 4H_{1,0} - 4\zeta_2 + 8\epsilon\Big[H_{1,1,0} + \zeta_2 H_1 - \zeta_3\Big] + \mathcal{O}(\epsilon^2)\,,$$
$$z_-\, I^{\mathrm{RR}}(0,1,0,0,-\epsilon,\epsilon,0,1,0) = \frac{H_0}{\epsilon} + H_{0,0} - \epsilon\Big[H_{0,0,0} + 2H_{0,1,0} + 2\zeta_2 H_0 + 4\zeta_3\Big] + \mathcal{O}(\epsilon^2)\,,$$
$$z_-\, I^{\mathrm{RR}}(0,1,0,0,-\epsilon,\epsilon,0,0,1) = \frac{H_0}{\epsilon} - H_{0,0} + \epsilon\Big[H_{0,0,0} + 2H_{0,1,0} + 2\zeta_2 H_0 + 4\zeta_3\Big] + \mathcal{O}(\epsilon^2)\,,$$

$$
\begin{aligned}
z_- z_+\, I^{\rm RR}(0,1,0,0,-\epsilon,\epsilon,1,0,1) =& \frac{2H_0}{\epsilon} - 4H_{-1,0} + 2H_{0,0} - 2\zeta_2 + 2\epsilon\Big[4H_{-1,-1,0} - 2H_{-1,0,0} \\
& - 4H_{0,-1,0} + H_{0,0,0} + 2H_{0,1,0} + 2\zeta_2 H_{-1} - \zeta_3\Big] + \mathcal{O}(\epsilon^2)\,, \\
z\, I^{\rm RR}(0,1,0,0,-\epsilon,\epsilon,1,1,0) =& -\frac{1}{\epsilon} + \frac{2zH_0}{z_-} + 2 + 2\epsilon\left[z\frac{H_{0,0} - 2H_0}{z_-} + H_{1,0} + \zeta_2 - 2\right] + \mathcal{O}(\epsilon^2)\,, \\
I^{\rm RR}(0,2,0,0,-\epsilon,\epsilon,1,1,0) =& \frac{1}{6\epsilon}\frac{z_-^2}{z^2} - \frac{z^2+z+10}{9z^2} + \frac{H_0}{3}\left[1 + \frac{2}{z_-}\right] + \frac{\epsilon}{9}\left[H_0\left(\frac{2z}{z_-} + \frac{3}{z}\right)\right. \\
& \left. + 3H_{0,0}\left(\frac{2}{z_-} + 1\right) - 3\big(H_{1,0} + \zeta_2\big)\frac{z_-^2}{z^2} + \frac{47}{3z^2} + \frac{11}{3z} + \frac{2}{3}\right] + \mathcal{O}(\epsilon^2)\,, \\
I^{\rm RR}(0,1,-1,0,1-\epsilon,\epsilon,0,0,1) =& \, 0 + \mathcal{O}(\epsilon)\,, \\
I^{\rm RR}(0,1,-1,0,1-\epsilon,\epsilon,0,1,0) =& \, 0 + \mathcal{O}(\epsilon)\,, \\
I^{\rm RR}(-1,2,-2,0,1-\epsilon,\epsilon,0,1,0) =& \, 0 + \mathcal{O}(\epsilon)\,,
\end{aligned}
$$

where again some integrals have been multiplied by $z_-$, $z_+$ or $z$.

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